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ELEMENTS OF THE THEORY OF THE NEWTONIAN POTENTIAL FUNCTION

Third, Revised and Enlarged Edition

BY

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RPI

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PREFACE TO THIRD EDITION.

This little text-book was written some years ago to accompany the lectures in a short preparatory course on the Newtonian Potential Function, especially intended for students who were afterwards to begin a systematic study of the Mathematical Theory of Electricity and Magnetism, with the help of some of the standard treatises on the subject.

In preparing the present edition a few imperative changes have been made in the plates, some sections have been introduced, and a large number of simple miscellaneous problems have been added at the end of the last chapter.

The reader who wishes to get a thorough knowledge of the properties of the Potential Function and of its applications, is referred to the works mentioned in the list given below. Most of those that had then been published I consulted and used in writing these notes, and from some which have appeared since the body of this book was electrotyped I have borrowed material for problems: many other problems I have taken from various college and university examination papers. I am indebted also to my colleagues, Professors Trowbridge, Byerly, E. H. Hall, Osgood, Sabine, M. Bôcher, and C. A. Adams for valuable criticisms and suggestions.

The slight use which I have made of developments in terms of Spherical Harmonics and Bessel's Functions is explained by the fact that students who use this book in Harvard University study at the same time Professor Byerly's admirable *Treatise on Fourier's Series, and Spherical, Cylindrical and Ellipsoidal Harmonics.*

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In the following pages the change made in a function u by giving to the independent variable x the arbitrary increment Δx, and keeping the other independent variables, if there are any, unchanged, is denoted by $\Delta_x u$. Similarly, $\Delta_y u$ and $\Delta_z u$ represent the increments of u due to changes respectively in y alone and in z alone. The total change in u due to simultaneous changes in all the independent variables is sometimes denoted by Δu; so that, if $u = f(x, y, z)$,

$$\Delta u = \frac{\Delta_x u}{\Delta x} \cdot \Delta x + \frac{\Delta_y u}{\Delta y} \cdot \Delta y + \frac{\Delta_z u}{\Delta z} \cdot \Delta z + \epsilon,$$

where ϵ is an infinitesimal of an order higher than the first. The partial derivatives, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$, are denoted, for convenience, by $D_x u$, $D_y u$, and $D_z u$, and the sign $\doteq$ placed between a variable and a constant is used to show that the former is to be made to approach the latter as its limit. In those cases where it is desirable to draw attention to the fact that a certain derivative is total, the differential notation $\frac{du}{dx}$ is used.

It is tacitly assumed that the physical quantities under consideration can be represented in the regions to which the theorems refer, by continuous point functions, having continuous derivatives of the orders which present themselves in the investigation in hand. In a few instances, as the reader will see, a theorem is predicated of analytic functions only, when so narrow a limitation is not required by the proof given.

SHORT LIST OF WORKS ON THE POTENTIAL FUNCTION AND ITS APPLICATIONS.

Bacharach: Abriss der Geschichte der Potentialtheorie.

Bedell and Crehore: Alternating Currents.

Betti: Teorica delle Forze Newtoniane e sue Applicazioni all' Elettrostatica e al Magnetismo. Also W. F. Meyer's translation of the same work into German.

Bôcher: Reihenentwicklungen der Potentialtheorie.

Boltzmann: Vorlesungen über Maxwell's Theorie der Elektricität und des Lichtes.

Burkhardt and Meyer: Die Potentialtheorie.

Chrystal: The article "Electricity" in the Ninth Edition of the Encyclopædia Britannica.

Clausius: Die Potentialfunction und das Potential.

Cumming: An Introduction to the Theory of Electricity.

Dirichlet: Vorlesungen über die im umgekehrten Verhältniss des Quadrats der Entfernung wirkenden Kräfte. Edited by Grube.

Drude: Physik des Aethers.

Duhem: Leçons sur l'Électricité et le Magnétisme.

Ewing: Magnetic Induction in Iron and other Metals.

Ferrers: Spherical Harmonics.

Fleming: The Alternate Current Transformer.

Franklin and Williamson: The Elements of Alternating Currents.

Gauss: Allgemeine Lehrsätze in Beziehung auf die im verkehrten Verhältnisse des Quadrates der Entfernung wirkenden Anziehungs- und Abstossungskräfte. Also other papers to be found in Volume V of his Gesammelte Werke.

Gray: Absolute Measurements in Electricity and Magnetism. A Treatise on Electricity and Magnetism.

Green: An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism.

Harnack: Grundlagen der Theorie des logarithmischen Potentiales.

Heaviside: Electrical Papers.

Heine: Kugelfunctionen.

Helmholtz: Wissenschaftliche Abhandlungen.

Hertz: Gesammelte Abhandlungen.

Jordan: Cours d'Analyse.

Joubert, Foster, and Atkinson: Elementary Treatise on Electricity and Magnetism.

Kirchhoff: Gesammelte Abhandlungen. Vorlesungen über mathematische Physik.

Elektricität und Magnetismus. Edited by Planck.

Klein: Vorlesungen über die Potentialtheorie.

Lamé: Leçons sur les Coordonnées Curvilignes et leurs Diverses Applications.

Mascart: Traité d'Électricité Statique. Also Wallentin's translation of the same work into German, with additions.

Mascart et Joubert: Leçons sur l'Électricité et le Magnétisme. Also Atkinson's translation of the same work into English, with additions.

Mathieu: Théorie du Potential et ses Applications à l'Électrostatique et au Magnétisme.

Maxwell: An Elementary Treatise on Electricity. A Treatise on Electricity and Magnetism.

Minchin: A Treatise on Statics.

Neumann, C.: Untersuchungen über das logarithmische und Newton'sche Potential.

Neumann, F.: Vorlesungen über die Theorie des Potentials und der Kugelfunctionen.

Nipher: Electricity and Magnetism.

Picard: Traité d'Analyse.

Poincaré: Électricité et Optique. Les Oscillations Électriques. Théorie du Potential Newtonien.

Riemann: Schwere, Electricität und Magnetismus. Edited by Hattendorff.

Routh: Analytical Statics.

Schell: Theorie der Bewegung und der Kräfte.

Steinmetz: Alternating Current Phenomena.

Tarleton: The Mathematical Theory of Attraction.

Thomson, J. J.: Elements of Electricity and Magnetism. Recent Researches in Electricity and Magnetism.

Thomson, W.: Reprint of Papers on Electrostatics and Magnetism.

Thomson and Tait: A Treatise on Natural Philosophy.

Todhunter: A History of the Mathematical Theories of Attraction and the Figure of the Earth. The Functions of Laplace, Bessel, and Lamé.

Turner: Examples on Heat and Electricity.

Watson and Burbury: The Mathematical Theory of Electricity and Magnetism.

Webster: The Theory of Electricity and Magnetism.

Wiedemann: Die Lehre von der Elektricität.

Winkelmann: Handbuch der Physik.

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THE
NEWTONIAN POTENTIAL FUNCTION.

CHAPTER I.

THE ATTRACTION OF GRAVITATION.

1. The Law of Gravitation. Every body in the universe attracts every other body with a force which depends for magnitude and direction upon the masses of the two bodies and upon their relative positions.

An *approximate* value of the attraction between any two rigid bodies may be obtained by imagining the bodies to be divided into small particles, and assuming that every particle of the one body attracts every particle of the other with a force directly proportional to the product of the masses of the two particles, and inversely proportional to the square of the distance between their centres or other corresponding points. The *true* value of the attraction is the limit approached by this approximate value as the particles into which the bodies are supposed to be divided are made smaller and smaller.

2. The Attraction at a Point. By "the attraction *at* any point P in space, due to one or more attracting masses," is meant the limit which would be approached by the value of the attraction on a sphere of unit mass centred at P if the radius of the sphere were made continually smaller and smaller while its mass remained unchanged. The attraction at P is, then, the attraction on a unit mass supposed to be *concentrated* at P.

If the attraction at every point throughout a certain region has a value other than zero, the region is called "a field of force"; and the attraction at any point P in the region is called "the strength of the field" at that point.

3. The Unit of Force. It will presently appear that all spheres made of homogeneous material attract bodies outside of themselves as if the masses of the spheres were concentrated at their middle points. If, then, k be the force of attraction between two unit masses concentrated at points at the unit distance apart, the attraction at a point P due to a homogeneous sphere of radius a and of density ρ is $k \cdot \frac{4\pi a^3 \rho}{3r^2}$, where r is the distance of P from the centre of the sphere. In all that follows, however, we shall take as our *unit of force* the force of attraction * between two unit masses concentrated at points at the unit distance apart. Using these units, k in the expression given above becomes 1, and the attraction between two particles of mass m_1 and m_2 concentrated at points r units apart is $\frac{m_1 m_2}{r^2}$.

4. Attraction due to Discrete Particles. The attraction at a point P, due to particles concentrated at different points in the same plane with P, may be expressed in terms of two components at right angles to each other.

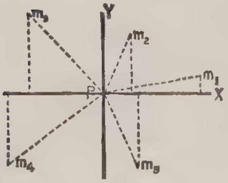

FIG. 1.

Let the straight lines joining P with the different particles be denoted by r_1, r_2, r_3, $\cdots$, and the angles which these lines make with some fixed line Px, by α_1, α_2, α_3, $\cdots$. If, then, the masses of the several particles are respectively $m_1, m_2, m_3, \cdots$, the components of the attraction at P are

---

* These are called "attraction units of force." When the attraction between two bodies is given in terms of absolute kinetic force units in any system, the corresponding value of k is sometimes called the "constant of gravitation." One dyne is equivalent to about 1.543×10^7 c.g.s. attraction units and one poundal to about 9.63×10^8 f.p.s. attraction units. For simple illustrative problems the reader may consult the Miscellaneous Examples at the end of the book.

$$X = \frac{m_1 \cos \alpha_1}{r_1^2} + \frac{m_2 \cos \alpha_2}{r_2^2} + \cdots = \sum \frac{m \cos \alpha}{r^2} \qquad [1]$$

in the direction Px, and

$$Y = \frac{m_1 \sin \alpha_1}{r_1^2} + \frac{m_2 \sin \alpha_2}{r_2^2} + \cdots = \sum \frac{m \sin \alpha}{r^2} \qquad [2]$$

in the direction Py, perpendicular to Px.

The resultant force at P is

$$R = \sqrt{X^2 + Y^2}, \qquad [3]$$

and its line of action makes with Px the angle whose tangent is $\frac{Y}{X}$.

If the particles do not all lie in the same plane with P, we may draw through P three mutually perpendicular axes, and call the angles which the lines joining P with the different particles make with the first axis $\alpha_1, \alpha_2, \alpha_3, \cdots$; with the second axis, $\beta_1, \beta_2, \beta_3, \cdots$; and with the third axis, $\gamma_1, \gamma_2, \gamma_3, \cdots$. The three components in the directions of these axes of the attraction at P due to all the particles are then

$$X = \sum \frac{m \cos \alpha}{r^2}; \quad Y = \sum \frac{m \cos \beta}{r^2}; \quad Z = \sum \frac{m \cos \gamma}{r^2}. \qquad [4]$$

The resultant attraction is

$$R = \sqrt{X^2 + Y^2 + Z^2}, \qquad [5]$$

and its line of action makes with the axes angles whose cosines are respectively

$$\frac{X}{R}, \ \frac{Y}{R}, \ \text{and} \ \frac{Z}{R}. \qquad [6]$$

5. Attraction at a Point in the Produced Axis of a Straight Wire. Let μ be the mass of the unit of length of a uniform straight wire AB of length l, and of cross section so small that we may suppose the mass of the wire concentrated in its axis (see Fig. 2), and let P be a point in the line AB produced at a

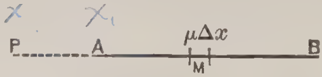

FIG. 2.

distance a from A. Divide the wire into elements of length Δx. The attraction at P due to one of these elements, M, whose nearest point is at a distance x from P, is less than $\frac{\mu\Delta x}{x^2}$ and greater than $\frac{\mu\Delta x}{(x+\Delta x)^2}$.

The attraction at P due to the whole wire lies between $\sum\frac{\mu\Delta x}{x^2}$ and $\sum\frac{\mu\Delta x}{(x+\Delta x)^2}$; but these quantities approach the same limit as Δx is made to approach zero, so that the attraction at P is

$$\operatorname*{limit}_{\Delta x \doteq 0}\sum\frac{\mu\Delta x}{x^2}=\int_a^{a+l}\frac{\mu\,dx}{x^2}=\mu\left[\frac{1}{a}-\frac{1}{a+l}\right]. \qquad [7]$$

If the coördinates of P, A, and B are respectively $(x, 0, 0)$, $(x_1, 0, 0)$, and $(x_1+l, 0, 0)$, this result may be put into the form

$$\mu\left[\frac{1}{x_1-x}-\frac{1}{x_1-x+l}\right]. \qquad [8]$$

6. Attraction at any Point, due to a Straight Wire. Let P (Fig. 3) be any point in the perpendicular drawn to the straight wire AB at A, and let $PA=c$, $AB=l$, $AM=x$, and the angle $ABP=\delta$. Let MN be one of the equal elements of mass $(\mu\Delta x)$ into which the wire is divided, and call PM, r. The attraction at P due to this element is approximately equal to $\frac{\mu\Delta x}{r^2}$, and acts in some direction lying between PM and PN. This attraction can be resolved into two components whose approximate values are $\frac{\mu\Delta x\cdot c}{(c^2+x^2)^{\frac{3}{2}}}$ in the direction PA, and $\frac{\mu\Delta x\cdot x}{(c^2+x^2)^{\frac{3}{2}}}$ in the direction PL. The true values of the components in these directions of the attraction at P, due to the whole wire, are, then, respectively:

$$\int_0^l \frac{\mu c\,dx}{(c^2+x^2)^{\frac{3}{2}}} = \frac{\mu}{c}\left[\frac{x}{\sqrt{c^2+x^2}}\right]_0^l = \frac{\mu}{c}\cos\delta, \qquad [9]$$

and

$$\int_0^l \frac{\mu x\,dx}{(c^2+x^2)^{\frac{3}{2}}} = \frac{-\mu}{c}\left[\frac{c}{\sqrt{c^2+x^2}}\right]_0^l = \frac{\mu}{c}(1-\sin\delta). \qquad [10]$$

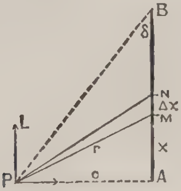

Fig. 3.

The resultant attraction is equal to the square root of the sum of the squares of these components, or

$$R = \frac{\mu}{c}\sqrt{2(1-\sin\delta)} = \frac{\mu}{c}\sqrt{2(1-\cos APB)} = \frac{2\mu}{c}\sin\tfrac{1}{2}APB, \quad [11]$$

and its line of action makes with PA an angle whose tangent is

$$\frac{1-\sin\delta}{\cos\delta} = \frac{1-\cos APB}{\sin APB} = \frac{2\sin^2\frac{1}{2}APB}{2\sin\frac{1}{2}APB\cdot\cos\frac{1}{2}APB} = \tan\tfrac{1}{2}APB.$$

That is, the resultant attraction at P acts in the direction of the bisector of the angle APB.

From these results we can easily obtain the value of the attraction at any point P, due to a uniform straight wire $B'B$ (Fig. 4). Drop a perpendicular PA from P upon the axis of the wire. Let $AB = l$, $AB' = l'$, $PA = c$, $ABP = \delta$, $AB'P = \delta'$, $BPB' = \theta$. The component in the direction PA of the attraction at P is [9]

$$\frac{\mu}{c}(\cos\delta + \cos\delta'),$$

and that in the direction PL is

$$\frac{\mu}{c}(\sin\delta' - \sin\delta) = \mu\left(\frac{1}{PB'} - \frac{1}{PB}\right),$$

so that the resultant attraction is

$$R = \frac{\mu}{c}\sqrt{2[1+\cos(\delta+\delta')]} = \frac{2\mu}{c}\cos\tfrac{1}{2}(\delta+\delta') = \frac{2\mu}{c}\sin\tfrac{1}{2}\theta. \quad [12]$$

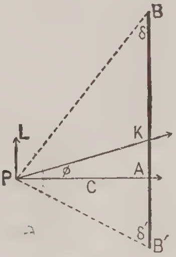

Fig. 4.

The line of action PK of R makes with PA an angle ϕ such that

$$\tan\phi = \frac{\sin\delta' - \sin\delta}{\cos\delta + \cos\delta'} = \tan\tfrac{1}{2}(\delta' - \delta); \qquad [13]$$

$$\therefore B'PK = \frac{\pi}{2} - \delta' + \tfrac{1}{2}(\delta' - \delta) = \frac{\pi}{2} - \tfrac{1}{2}(\delta + \delta'),$$

and

$$BPK = \frac{\pi}{2} - \delta - \tfrac{1}{2}(\delta' - \delta) = \frac{\pi}{2} - \tfrac{1}{2}(\delta + \delta').$$

It is to be noticed that PK bisects the angle θ, and does not in general pass through the centre of gravity or any other fixed point of the wire. Indeed, the path of a particle moving from rest under the attraction of a straight wire is generally curved; for if the particle should start at a point Q and move a short distance on the bisector of the angle BQB' to Q', the attraction of the wire would now urge the particle in the direction of the bisector of the angle $BQ'B'$, and this is usually not coincident with the bisector of BQB'.

If q is the area of the cross section of the wire, and ρ the mass of the unit volume of the substance of which the wire is made, we may substitute for μ in the formulas of this section its value $q\rho$.

If instead of a very thin wire we had a body in the shape of a prism or cylinder of considerable cross section, we might divide this up into a large number of slender prisms and use the equations just obtained to find the limit of the sum of the attractions at any point due to all these elementary prisms. This would be the attraction due to the given body.

7. Attraction at a Point in the Produced Axis of a Cylinder of Revolution. In order to find the attraction due to a homogeneous cylinder of revolution at any point P (Fig. 5) in the axis of the cylinder produced, it will be convenient to imagine the cylinder cut up into discs of constant thickness Δc, by means of planes perpendicular to the axis.

Let ρ be the mass of the unit of volume of the cylinder, and a the radius of its base. Consider a disc whose nearer face is at a distance c from P, and divide it into elements by means of radial planes drawn at angular intervals of $\Delta\theta$ and concentric cylindrical surfaces at radial intervals of Δr.

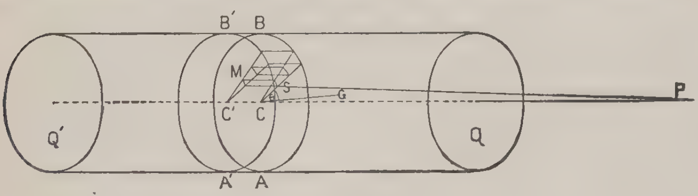

FIG. 5.

The mass of any element M whose inner radius is r is equal to $\rho\Delta c\cdot\Delta\theta[r\Delta r+\frac{1}{2}(\Delta r)^2]$, and the whole attraction at P due to M is approximately $\rho\dfrac{\Delta\theta\Delta c[r\Delta r+\frac{1}{2}(\Delta r)^2]}{c^2+r^2}$ in a line joining P with some point of M. The component of this attraction in the direction PC is found by multiplying the expression just given by $\frac{c}{\sqrt{c^2+r^2}}$, the cosine of the angle CPS, so that the attraction at P in the direction PC, due to the whole disc, is approximately

$$\Delta c \cdot \lim \sum \frac{\rho c\, \Delta\theta \left[r\Delta r + \frac{1}{2}(\Delta r)^2\right]}{(c^2+r^2)^{\frac{3}{2}}} = \Delta c \int_0^{2\pi} d\theta \int_0^a \frac{\rho c r\, dr}{(c^2+r^2)^{\frac{3}{2}}}$$

$$= 2\pi\rho\,\Delta c\left[1 - \frac{c}{\sqrt{c^2+a^2}}\right]. \qquad [14]$$

If the bases of the cylinder are at distances c_0 and c_0+h from P, the true value of the attraction at P in the direction PC, due to the cylinder QQ', is

$$\underset{\Delta c \doteq 0}{\text{limit}} \sum 2\pi\rho\,\Delta c\left[1 - \frac{c}{\sqrt{c^2+a^2}}\right] = 2\pi\rho \int_{c_0}^{c_0+h} \left(1 - \frac{c}{\sqrt{c^2+a^2}}\right) dc$$

$$= 2\pi\rho\left[h + \sqrt{c_0^2+a^2} - \sqrt{(c_0+h)^2+a^2}\right]. \qquad [15]$$

This is evidently the whole attraction at P due to the cylinder, for considerations of symmetry show us that the resultant attraction at P has no component perpendicular to PC.

[14] gives the attraction due to the elementary disc $ABA'B'$, on the assumption that the whole matter of the disc is concentrated at the face ABC. The actual attraction at P due to this disc may be found by putting $c_0 = c$ and $h = \Delta c$ in [15].

If a, the radius of the cylinder, is very large compared with h and c_0, the expression [15] for the attraction at P due to the cylinder approaches the value $2\pi\rho h$.

8. Attraction at the Vertex of a Cone. The attraction due to a homogeneous cone of revolution, at a point at the vertex of the cone, may be found by the aid of [14].

If Fig. 6 represents a plane section of the cone taken through the axis, and if $PM = c$, $MM' = \Delta c$, and $MB = r$, the attraction at P due to the disc $ABCD$ is approximately

$$2\pi\rho\,\Delta c\left[1 - \frac{c}{\sqrt{c^2+r^2}}\right] = 2\pi\rho\,\Delta c\,(1 - \cos\alpha),$$

and the attraction due to the whole cone is

$$\underset{\Delta c \doteq 0}{\text{limit}} \sum 2\pi\rho(1-\cos\alpha)\Delta c = 2\pi\rho(1-\cos\alpha) \underset{\Delta c \doteq 0}{\text{limit}} \sum \Delta c$$
$$= 2\pi\rho(1-\cos\alpha)\cdot PL. \qquad [16]$$

The attraction at P due to the frustum $ABKN$ is found by subtracting the value of the attraction due to the cone ABP from the expression given in [16]. The result is

$$2\pi\rho(1-\cos\alpha)(PL-PM) = 2\pi\rho(1-\cos\alpha)ML, \qquad [17]$$

and it is easy to see from this that discs of equal thickness cut out of a cone of revolution at different distances from the vertex by planes perpendicular to the axis exert equal attractions at the vertex of the cone.

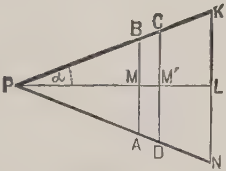

Fig. 6.

It follows almost directly that the portions cut out of two concentric spherical shells of equal uniform density and equal thickness, by *any* conical surface having its vertex at the common centre P of the shells, exert equal attraction at this centre; but we may prove this proposition otherwise, as follows:

Divide the inner surface of the portion cut out of one of the shells by the given cone into elements, and make the perimeter of each of these surface elements the directrix of a conical surface having its vertex at P. Divide the given shells into elementary shells of thickness Δr by means of concentric spherical surfaces drawn about P. In this way the attracting masses will be cut up into volume elements.

Let ML' (Fig. 7) represent one of these elements, whose inner surface has a radius equal to r; then, if the elementary cone APB intercept an element of area $\Delta\omega$ from a spherical surface of radius unity drawn around P, the area of the surface element at MM' is $r^2\Delta\omega$, and that at LL' is $(r+\Delta r)^2\Delta\omega$. The attraction at P in the direction PM, due to the element ML', is approximately

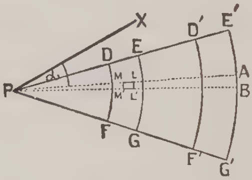

FIG. 7.

$$\rho\frac{r^2\Delta\omega\Delta r}{r^2}=\rho\Delta\omega\Delta r,$$

and the component of this in any direction Px, making an angle α with PM, is approximately $\rho\Delta\omega\Delta r\cos\alpha$. The attraction at P in the direction Px, due to the whole shell $EDFG$, is, then,

$$X=\lim\sum\rho\Delta r\Delta\omega\cos\alpha,$$

where the sum is to include all the volume elements which go to make up the shell. If $PF=r_0$, $PG=r_1$, $PF'=r_0'$, $PG'=r_1'$, and $\mu=FG=F'G'$,

$$X=\int_{r_0}^{r_1}\rho dr\int\cos\alpha\,d\omega=\rho\mu\int\cos\alpha\,d\omega.$$

The attraction at P in the same direction, due to the shell $E'D'F'G'$, is

$$X'=\rho\int_{r_0'}^{r_1'}dr\int\cos\alpha\,d\omega=\rho\mu\int\cos\alpha\,d\omega.$$

But the limits of integration with regard to ω are the same in both cases; $\therefore$ $X=X'$, which was to be proved.

If the shells are of different thicknesses, it is evident that they will exert attractions at P proportional to these thicknesses.

The area of the portion which a conical surface cuts out of a spherical surface of unit radius drawn about the vertex of the cone is called "the solid angle" of the conical surface.

9. Attraction of a Spherical Shell. In order to find the attraction at P, any point in space, due to a homogeneous spherical shell of radii r_0 and r_1, it will be best to begin by dividing up the shell into a large number of concentric shells of thickness Δr, and to consider first the attraction of one of these thin shells, whose inside radius shall be r.

Let ρ be the density of the given shell, that is, the mass of the unit of volume of the material of which the shell is composed. Join P (Fig. 8) with O by a straight line cutting the inner surface of the thin shell at N, and pass a plane through PO cutting this inner surface in a great circle $NLSL'$, which will serve as a prime meridian. Using N as a pole, describe upon the inner surface of the thin shell a number of parallels of latitude so as to cut off equal arcs on $NLSL'$. Denote by $\Delta\theta$ the angle which each one of these arcs subtends at O. Through PO pass a number of planes so as to cut up each parallel of latitude into equal arcs. Denote by $\Delta\phi$ the angle between any two contiguous planes of this series. By this means the inner surface of the elementary shell will be divided into small quadrilaterals, each of which will have two sides formed of meridian arcs, of length $r \cdot \Delta\theta$, and two sides formed of arcs of parallels of latitude, of length $r \sin\theta \cdot \Delta\phi$ and $r \sin(\theta + \Delta\theta) \cdot \Delta\phi$, where

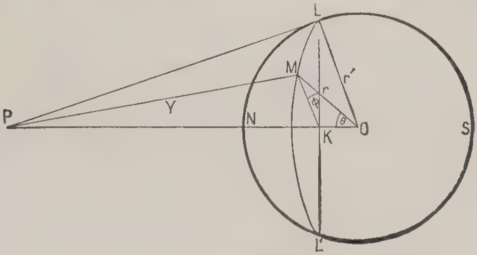

FIG. 8.

θ is the angle which the radius drawn to the parallel of higher latitude makes with ON. The area of one of these quadrilaterals is approximately $r^2 \sin\theta \cdot \Delta\theta \cdot \Delta\phi$, and the thickness of the shell is Δr, so that the element of volume is approximately $r^2 \sin\theta \cdot \Delta r \cdot \Delta\theta \cdot \Delta\phi$. Let $PM = y$, then the attraction at P, due to an element of mass which has a corner at M, is approximately $\dfrac{\rho r^2 \sin\theta\,\Delta r\,\Delta\theta\,\Delta\phi}{y^2}$, in the direction PM. This force may be resolved into three components: one in the direction PO, the others in directions perpendicular to PO and to each other; but it is evident from considerations of symmetry that in finding the attraction at P due to the whole shell we shall need only that component which acts in PO. This is approximately $\dfrac{\rho r^2 \sin\theta \cdot \Delta r\,\Delta\theta\,\Delta\phi \cdot \cos KPM}{y^2}$; or, if $PO = c$,

$$\frac{\rho r^2 \sin\theta (c - r\cos\theta)\,\Delta r\,\Delta\theta\,\Delta\phi}{y^3}. \qquad [18]$$

The attraction at P due to the whole elementary shell is, then, approximately (truly on the assumption that the whole mass of the shell is concentrated at its inner surface),

$$\Delta r \int\!\!\int \frac{\rho r^2 \sin\theta\,(c - r\cos\theta)\,d\theta\,d\phi}{y^3} = \Delta r\,X; \qquad [19]$$

and the true value at P of the attraction due to the given shell is

$$\int_{r_0}^{r_1} X\,dr. \qquad [20]$$

If in the expression for X we substitute for θ its value in terms of y, we have, since

$$y^2 = c^2 + r^2 - 2\,cr\cos\theta,$$

and hence $\qquad 2\,y\,dy = 2\,cr\sin\theta\,d\theta,$

$$X = \int_0^{2\pi} d\phi \int_{y_0}^{y_1} \frac{\rho r\,dy}{2\,c^2 y^2}(c^2 - r^2 + y^2) = \frac{\pi\rho r}{c^2}\int_{y_0}^{y_1}\left(\frac{c^2 - r^2}{y^2} + 1\right)dy$$

$$= \frac{\pi\rho r}{c^2}\left[\frac{r^2 - c^2 + y^2}{y}\right]_{y_0}^{y_1}. \qquad [21]$$

In order to find the limits of the integration with regard to y, we must distinguish between two cases:

I. If P is a point in the cavity enclosed by the given shell,

$$y_0 = r - c \quad \text{and} \quad y_1 = r + c;$$

$$\therefore\ X = \frac{\pi \rho r}{c^2}\left[\frac{r^2 - c^2 + (r+c)^2}{r+c} - \frac{r^2 - c^2 + (r-c)^2}{r-c}\right] = 0, \quad [22]$$

and

$$\int_{r_0}^{r_1} X dr = 0; \quad [23]$$

so that a homogeneous spherical shell exerts no attraction at points in the cavity which it encloses.

II. If P is a point without the given shell,

$$y_0 = c - r \quad \text{and} \quad y_1 = c + r;$$

$$\therefore\ X = \frac{\pi \rho r}{c^2}\left[\frac{r^2 - c^2 + (c+r)^2}{c+r} - \frac{r^2 - c^2 + (c-r)^2}{c-r}\right] = \frac{4\pi\rho r^2}{c^2}, \quad [24]$$

and

$$\int_{r_0}^{r_1} X dr = \frac{4}{3}\frac{\pi\rho}{c^2}(r_1^3 - r_0^3). \quad [25]$$

From this it follows that the attraction due to a spherical shell of uniform density is the same, at a point without the shell, as the attraction due to a mass equal to that of the shell concentrated at the shell's centre.

If in [25] we make $r_0 = 0$, we have the attraction, due to a solid sphere of radius r_1 and density ρ, at a point outside the sphere at a distance c from the centre. This is

$$\frac{4\pi\rho r_1^3}{3c^2}. \quad [26]$$

10. Attraction due to a Hemisphere. At any point P in the plane of the base of a homogeneous hemisphere, the attraction of the hemisphere gives rise to two components, one directed toward the centre of the base, the other perpendicular to the plane of the base. We will compute the values of these components for the particular case where P lies on the rim of the hemisphere's base, and for this purpose we will take the origin of our system of polar coördinates at P, because by so doing we shall escape having to deal with a quantity which becomes infinite at one of the limits of integration. Denote the coördinates of any point L in the hemisphere by r, θ, ϕ, where (Fig. 9) $XPN = \phi$, $IPL = \theta$, and $PL = r$.

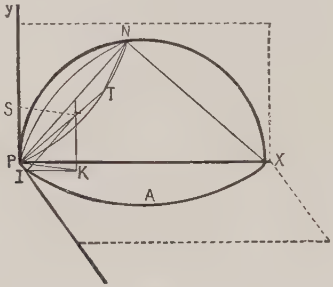

Fig. 9.

If r_1 be the radius of the hemisphere,

$$PT = PN \cos NPT = PX \cos XPN \cdot \cos NPT = 2\, r_1 \sin\theta \cos\phi.$$

$$\cos XPL = \frac{IK}{PL} = \frac{IK}{r} = \frac{IL \cos\phi}{r} = \sin\theta \cos\phi.$$

$$\cos SPL = \frac{PS}{PL} = \frac{KL}{r} = \frac{IL \sin\phi}{r} = \sin\theta \sin\phi.$$

The mass of a polar element of volume whose corner is at L is approximately $\rho \cdot IL\Delta\phi \cdot PL\Delta\theta \cdot \Delta r$ or $\rho r^2 \sin\theta\, \Delta r\, \Delta\theta\, \Delta\phi$, and this divided by r^2 is the attraction at P in the direction PL of the element, supposed concentrated at L. The components of this attraction in the direction PX and PY are respectively $\rho \sin\theta\, \Delta r\, \Delta\theta\, \Delta\phi \cos XPL$ and $\rho \sin\theta\, \Delta r\, \Delta\theta\, \Delta\phi \cos SPL$.

The component in the direction Py of the attraction at P due to the whole hemisphere is, then,

$$\int_0^{\frac{\pi}{2}} d\phi \int_0^{\pi} d\theta \int_0^{2\, r_1 \sin\theta \cos\phi} \rho \sin^2\theta \sin\phi\, dr = \tfrac{4}{3}\rho r_1, \qquad [27]$$

and the component in the direction Px is

$$\int_0^{\frac{\pi}{2}} d\phi \int_0^{\pi} d\theta \int_0^{2r_1\sin\theta\cos\phi} \rho\sin^2\theta\cos\phi\,dr = \tfrac{2}{3}\pi\rho r_1. \qquad [28]$$

This last expression might have been obtained from [26] by making c equal to r and halving the result.

11. Attraction of a Hemispherical Hill. If at a point on the earth at the southern extremity of a homogeneous hemispherical hill of density ρ and radius r_1 the force of gravity due to the earth, supposed spherical, is g, the attraction due to the earth and the hill will give rise to two components, $g-\frac{4}{3}\rho r_1$ downwards, and $\frac{2}{3}\pi\rho r_1$ northwards. The resultant attraction does not therefore act in the direction of the centre of the earth, but makes with this direction an angle whose tangent is $\dfrac{\frac{2}{3}\pi\rho r_1}{g-\frac{4}{3}\rho r_1}$.

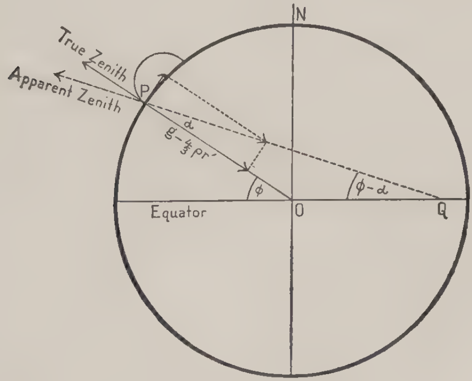

FIG. 10.

Let ϕ (Fig. 10) be the true latitude of the place and $(\phi-\alpha)$ the apparent latitude, as obtained by measuring the angle which the plumb-line at the place makes with the plane of the equator. Let a be the radius of the earth and σ its average density. Then

$$\tan\alpha = \frac{\frac{2}{3}\pi\rho r_1}{g-\frac{4}{3}\rho r_1} = \frac{\pi\rho r_1}{2(\pi a\sigma-\rho r_1)}. \qquad [29]$$

The radius of the earth is very large compared with the radius of the hill, and α is a small angle, so that approximately $\alpha = \frac{\rho r_1}{2\,a\sigma}$, and the apparent latitude of the place is $\phi - \frac{\rho r_1}{2\,a\sigma}$.

If ϕ_1 is the true latitude of a place just north of the same hill, its apparent latitude will be $\phi_1 + \frac{\rho r_1}{2\,a\sigma}$, and the apparent difference of latitude between the two places, one just north of the hill and the other just south of it, will be the true difference plus $\frac{\rho r_1}{a\sigma}$. If there were a hemispherical cavity between the two places instead of a hemispherical hill, the apparent difference of latitude would be less than the true difference.

12. Ellipsoidal Homœoids. A shell, thick or thin, bounded by two ellipsoidal surfaces, concentric, similar, and similarly placed, shall be called an *ellipsoidal homœoid*.

It is a property of every such shell that if any straight line cut its outer surface at the points S, S' (Fig. 11) and its inner surface at Q, Q', so that these four points lie in the order $SQQ'S'$, the length SQ will be equal to the length $Q'S'$.*

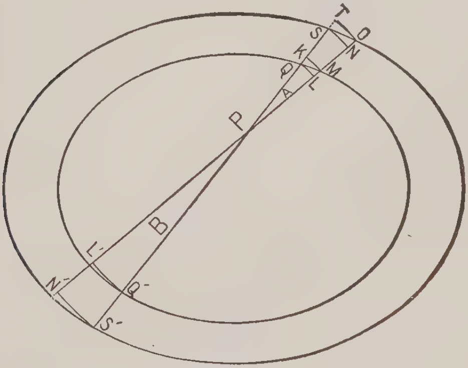

Fig. 11.

We will prove that the attraction of a homogeneous closed ellipsoidal homœoid, at any point P in the cavity which it shuts in, is zero.

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* The section of the homœoid made by a plane which passes through the centre and the secant line, is bounded by two concentric, similar, and similarly placed ellipses. This figure may be regarded as an orthogonal projection of two concentric circles cut by a straight line.

Make P the vertex of a slender conical surface of two nappes, A and B, and suppose the plane of the paper to be so chosen that PQ is the shortest and PM the longest length cut from any element of the nappe A by the inner surface of the homœoid. Draw about P spherical surfaces of radii PQ, PM, PS, and PO, and imagine the space between the innermost and outermost of these surfaces filled with matter of the same density as the homœoid. The nappe A cuts out a portion from this spherical shell whose trace on the plane of the paper is $QLOT$. Let us call this, for short, "the element $QLOT$." The attraction at P, due to the element $QMOS$ which A cuts out of the homœoid, is less than the attraction at the same point due to the element $QLOT$, and greater than that due to the element whose trace is $KMNS$. But the attraction at P, due to the first of these elements of spherical shells, is to the attraction due to the other as the thickness of the first shell is to that of the other, or as QT is to KS. (See Section 8.) The limit of the ratio of QT to KS, as the solid angle of the cone is made smaller and smaller, is unity; therefore the limit of the ratio of the attraction at P due to the element $QMOS$, to the attraction due to the element of spherical shell whose trace is $QLNS$, is unity. By a similar construction it is easy to show that the limit of the ratio of the attraction at P, due to the element which B cuts out of the homœoid, to the attraction due to the portion of spherical shell whose trace is $Q'L'N'S'$, is unity.

But the attractions at P, due to the elements $Q'L'N'S'$ and $QLNS$, are equal in amount (since their thicknesses are the same) and opposite in direction, so that if for the elements of the homœoid these elements were substituted, there would be no resultant attraction at P. In order to get the attraction at P in any direction due to the whole homœoid we may cut up the inner surface of the homœoid into elements, use the perimeter of each one of these elements as the directrix of a conical surface having its vertex at P, and find the limit of the sum of the attractions due to the elements which these conical surfaces cut from the homœoid. Wherever we have to find the finite limit of the sum of a series of infinitesimal quantities, we may without error substitute for any one of these another infinitesimal, the limit of whose ratio to the first is unity. For the attractions at P due to the elements of the homœoid we may, therefore, substitute attractions due to elements of spherical shells, which, as we have seen, destroy each other in pairs. Hence our proposition.

A shell bounded by two concentric spherical surfaces gives a special case under this theorem.

13. Sphere of Variable Density. The density of a homogeneous body is the amount of matter contained in the unit volume of the material of which the body is composed, and this may be obtained by dividing the mass of the body by its volume.

If the amount of matter contained in a given volume is not the same throughout a body, the body is called heterogeneous, and its density is said to be variable.

The average density of a heterogeneous body is the ratio of the mass of the body to its volume. The actual density ρ at any point Q inside the body is defined to be the limit of the ratio of the mass of a small portion of the body taken about Q to the volume of this portion as the latter is made smaller and smaller.

The attraction, at any point P, due to a spherical shell whose density is the same at all points equidistant from the common centre of the spherical surfaces which bound the shell but different at different distances from this centre, may be obtained with the help of some of the equations in Article 9.

Since ρ is independent of θ and ϕ, it may be taken out from under the signs of integration with regard to these variables, although it must be left under the sign of integration with regard to r.

Equations 19 to 24 inclusive hold for the case that we are now considering as well as for the case when ρ is constant, so that the attraction at all points within the cavity enclosed by a spherical shell whose density varies with the distance from the centre is zero.

If P is without the shell, the attraction is

$$\int_{r_0}^{r_1} X dr = \int_{r_0}^{r_1} \frac{4\pi\rho r^2 dr}{c^2},$$

or, if $\rho = f(r)$,

$$\frac{4\pi}{c^2}\int_{r_0}^{r_1} f(r)\cdot r^2 dr. \qquad [30]$$

The mass of the shell is evidently

$$\underset{\Delta r \doteq 0}{\text{limit}} \sum_{r_0}^{r_1} 4\pi r^2 \cdot f(r) dr = 4\pi \int_{r_0}^{r_1} f(r)\cdot r^2 dr, \qquad [31]$$

and [30] declares that a spherical shell whose density is a function of the distance from its centre attracts at all outside points as if the whole mass of the shell were concentrated at the centre.

If $r_0 = 0$, we have the case of a solid sphere.

14. Attraction due to any Mass. In order to find the attraction at a point P (Fig. 12), due to any attracting masses M', we may choose a system of rectangular coördinate axes and divide

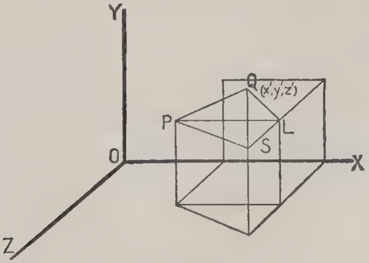

FIG. 12.

M' up into volume elements. If ρ is the average density of one of these elements $(\Delta v')$, the mass of the element will be $\rho\Delta v'$. Let Q, whose coördinates are x', y', z', be a point of the element, and let the coördinates of P be x, y, z. The attraction at P in the direction PQ due to this element is approximately $\frac{\rho\Delta v'}{\overline{PQ}^2}$, and the components of this in the direction of the coördinate axes are

$$\frac{\rho\Delta v'}{\overline{PQ}^2}\cos\alpha', \quad \frac{\rho\Delta v'}{\overline{PQ}^2}\cos\beta', \quad \text{and} \quad \frac{\rho\Delta v'}{\overline{PQ}^2}\cos\gamma', \qquad [32]$$

where α', β', γ' are the angles which PQ makes with the positive directions of the axes.

It is easy to see that

$$\cos\alpha' = \frac{PL}{PQ} = \frac{x'-x}{PQ},$$

and, similarly, that

$$\cos\beta' = \frac{y'-y}{PQ}, \quad \text{and} \quad \cos\gamma' = \frac{z'-z}{PQ}.$$

Moreover,

$$\overline{PQ}^2 = \overline{PL}^2 + \overline{LS}^2 + \overline{SQ}^2 = (x'-x)^2 + (y'-y)^2 + (z'-z)^2,$$

and this we will call r^2.

The true values of the components in the direction of the coördinate axes of the attraction at P, due to all the elements which go to make up M', are, then,

$$X = \underset{\Delta v' \doteq 0}{\text{limit}} \sum \frac{\rho\Delta v'(x'-x)}{r^3}$$

$$= \iiint \frac{\rho(x'-x)\,dx'dy'dz'}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{\frac{3}{2}}}; \qquad [33_A]$$

$$Y = \underset{\Delta v' \doteq 0}{\text{limit}} \sum \frac{\rho\Delta v'(y'-y)}{r^3}$$

$$= \iiint \frac{\rho(y'-y)\,dx'dy'dz'}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{\frac{3}{2}}}, \qquad [33_B]$$

$$Z = \underset{\Delta v' \doteq 0}{\text{limit}} \sum \frac{\rho\Delta v'(z'-z)}{r^3}$$

$$= \iiint \frac{\rho(z'-z)\,dx'dy'dz'}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{\frac{3}{2}}}; \qquad [33_C]$$

where ρ is the density at the point (x', y', z'), and where the integrations with regard to x', y', and z' are to include the whole of M'.

The resultant attraction at P, due to M', is

$$R = \sqrt{X^2 + Y^2 + Z^2}; \qquad [34]$$

and its line of action makes with the coördinate axes angles whose cosines are

$$\lambda = \frac{X}{R}, \quad \mu = \frac{Y}{R}, \quad \text{and} \quad \nu = \frac{Z}{R}. \qquad [35]$$

The component of the attraction at the point (x, y, z) in a direction making an angle ϵ with the line of action of R is $R \cos\epsilon$. If the direction cosines of this direction are λ', μ', ν', we have

$$\cos\epsilon = \lambda\lambda' + \mu\mu' + \nu\nu'.$$

15. The quantities X, Y, Z, and R, which occur in the last section, are in general functions of the coördinates x, y, and z of the point P. Let us consider X, whose value is given in $[33_A]$. If P lies without the attracting mass M', the quantity $\frac{x'-x}{r^3}$ is finite for all the elements into which M' is divided. Let L be the largest value which it can have for any one of these elements, then X is less than $L\int\int\int \rho\, dx'dy'dz'$, or $L \cdot M'$, and this is finite. If P is a point within the space which the attracting mass occupies, it is easy to show that, whatever physical meaning we may attach to X, it has a finite value. To prove this, make P the origin of a system of polar coördinates, and divide M' up into elements like those used in Section 10. It will then be clear that

$$X = \int\int\int \rho \sin^2\theta \cos\phi \, dr\, d\theta\, d\phi, \qquad [36]$$

where the limits are to be chosen so as to include all the attracting mass. Since $\sin^2\theta \cos\phi$ can never be greater than unity, X is less than $\int\int\int \rho \, dr \, d\theta \, d\phi$, which is evidently finite when ρ is finite, as it always is in fact.

The corresponding expressions,

$$Y = \int\int\int \rho \sin^2\theta \sin\phi \, dr \, d\theta \, d\phi, \tag{37}$$

and

$$Z = \int\int\int \rho \sin\theta \cos\theta \, dr \, d\theta \, d\phi, \tag{38}$$

can be proved finite in a similar manner; and it follows that X, Y, Z, and consequently R, are finite for all values of x, y, and z.

As a special case, the attraction at a point P within the mass of a homogeneous spherical shell, of radii r_0 and r_1, and of density ρ, is

$$\tfrac{4}{3}\pi\rho\left(\frac{r^3 - r_0^3}{r^2}\right), \tag{39}$$

where r is the distance of P from the centre of the shell.

16. Attraction between Two Straight Wires. Let AK and BK' (Fig. 13) be two straight wires of lengths l and l' and of line-densities μ and μ'; and let $KB = c$. Divide AK into elements of length Δx, and consider one of these MM', such that $AM = x$. The attraction of BK' on a unit mass concentrated at M would be (Sections 2 and 5), $\mu'\left[\frac{1}{MB} - \frac{1}{MK'}\right]$. If, therefore, the whole element MM' whose mass is $\mu\Delta x$ were concentrated at M, the attraction on it, due to BK', would be

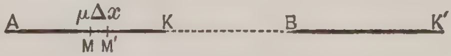

FIG. 13.

$$\mu\mu'\Delta x\left[\frac{1}{MB} - \frac{1}{MK'}\right] = \mu\mu'\Delta x\left[\frac{1}{l + c - x} - \frac{1}{l + l' + c - x}\right]. \tag{40}$$

The actual force, due to the attraction of BK', with which the whole wire AK is urged toward the right, is

$$\underset{\Delta x \doteq 0}{\text{limit}} \sum_0^l \mu\mu'\Delta x\left[\frac{1}{l+c-x} - \frac{1}{l+l'+c-x}\right]$$

$$= \mu\mu'\int_0^l\left(\frac{1}{x-(l+l'+c)} - \frac{1}{x-(l+c)}\right)dx$$

$$= \mu\mu'\left[\log\frac{x-l-l'-c}{x-l-c}\right]_0^l = \mu\mu'\log\frac{(l+c)(l'+c)}{c(l+l'+c)}. \qquad [41]$$

17. Attraction between Two Spheres. Consider two homogeneous spheres of masses M and M' (Fig. 14), whose centres C and C' are at a distance c from each other. Divide the sphere M' into elements in the manner described in Section 9. The attraction due to M at any point P' outside of this sphere is, as we have seen, $\frac{M}{\overline{CP'}^2}$, and its line of action is in the direction $P'C$.

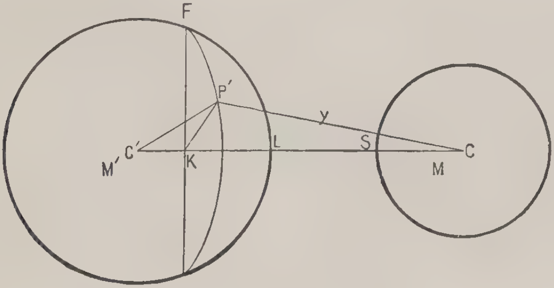

Fig. 14.

Let $P' = (r, \theta, \phi)$ be any point in the sphere M', and let $CP' = y$. The attraction of M in the direction $P'C$ on an element of mass $\rho r^2 \sin\theta \Delta r \Delta\theta \Delta\phi$ supposed concentrated at P' is $\frac{M\rho r^2 \sin\theta\, \Delta r\, \Delta\theta\, \Delta\phi}{y^2}$, and the component of this parallel to the line $C'C$ is $\frac{M\rho r^2 \sin\theta(c - r\cos\theta)\Delta r\,\Delta\theta\,\Delta\phi}{y^3}$. The force with which the whole sphere M' is urged toward the right by the attraction of M is, then,

$$M\int\int\int\frac{\rho r^2 \sin\theta\, dr\, d\theta\, d\phi\,(c - r\cos\theta)}{y^3}, \qquad [42]$$

where the integration is to be extended to all the elements which go to make up M'. It is proved in Section 9 that the value of this triple integral is $\frac{M'}{c^2}$, so that the force of attraction between the two spheres is $\frac{MM'}{c^2}$.

18. Attraction between any Two Rigid Bodies. In order to find the force with which a rigid body M is pulled in any direction (as for instance in that of the axis of x) by the attraction of another body M', we must in general find the value of a sextuple integral.

Let M be divided up into small portions, and let Δm be the mass of one of these elements which contains the point (x, y, z)

The component in the direction of the axis of x of the attraction at (x, y, z) due to M' is

$$\int\int\int\frac{\rho(x'-x)\,dx'dy'dz'}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{\frac{3}{2}}},$$

and this would be the actual attraction in this direction on a unit mass supposed concentrated at (x, y, z). If the mass Δm were concentrated at this point, the attraction on it in the direction of the axis of x would be

$$\Delta m\int\int\int\frac{\rho(x'-x)\,dx'dy'dz'}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{\frac{3}{2}}}. \qquad [43]$$

The actual attraction in the direction of the axis of x of M' upon the whole of M is, then,

$$\underset{\Delta m \doteq 0}{\text{limit}}\sum \Delta m\cdot\int\int\int\frac{\rho(x'-x)\,dx'dy'dz'}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{\frac{3}{2}}}. \qquad [44]$$

If ρ' is the density at the point (x, y, z), and if the elements into which M is divided are rectangular parallelopipeds of dimensions Δx, Δy, and Δz, the expression just given may be written

$$\int\int\int\int\int\int \frac{\rho'\rho(x'-x)\,dx\,dy\,dz\,dx'dy'dz'}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{\frac{3}{2}}}, \qquad [45]$$

where the integrations are first to be extended over M' and then over M.

EXAMPLES.

1. Find the resultant attraction, at the origin of a system of rectangular coördinates, due to masses of 12, 16, and 20 units respectively, concentrated at the points (3, 4), (−5, 12), and (8, −6). What is its line of action?

2. Find the value, at the origin of a system of rectangular coördinates, of the attraction due to three equal spheres, each of mass m, whose centres are at the points $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$. Find also the direction-cosines of the line of action of this resultant attraction.

3. Show that the attraction, due to a uniform wire bent into the form of the arc of a circumference, is the same at the centre of the circumference as the attraction due to any uniform straight wire of the same density which is tangent to the given wire, and is terminated by the bounding radii (when produced) of the given wire.

4. Show that in the case of an oblique cone whose base is any plane figure the attraction at the vertex of the cone due to any frustum varies, other things being equal, as the thickness of the frustum.

5. Find the equation of a family of surfaces over each one of which the resultant force of attraction due to a uniform straight wire is constant.

6. Using the foot-pound-second system of fundamental units, and assuming that the average density of the earth is 5.6, compare with the poundal the unit of force used in this chapter.

7. If in Fig. 2 we suppose P moved up to A, the attraction at P becomes infinite according to [7], and yet Section 15 asserts that the value, at any point inside a given mass, of the attraction due to this mass is always finite. Explain this.

8. A spherical cavity whose radius is r is made in a uniform sphere of radius $2r$ and mass m in such a way that the centre of the sphere lies on the wall of the cavity. Find the attraction due to the resulting solid at different points on the line joining the centre of the sphere with the centre of the cavity.

9. A uniform sphere of mass m is divided into halves by the plane AB passed through its centre C. Find the value of the attraction due to each of these hemispheres at P, a point on the perpendicular erected to AB at C, if $CP = a$.

10. Considering the earth a sphere whose density varies only with the distance from the centre, what may we infer about the law of change of this density if a pendulum swing with the same period on the surface of the earth and at the bottom of a deep mine? What if the force of attraction increases with the depth at the rate of $\frac{1}{n}$th of a dyne per centimetre of descent?

11. The attraction due to a cylindrical tube of length h and of radii R_0 and R_1, at a point in the axis, at a distance c_0 from the plane of the nearer end, is

$$2\pi\rho\left[\sqrt{c_0^2+R_1^2}-\sqrt{c_0^2+R_0^2}+\sqrt{(c_0+h)^2+R_0^2}-\sqrt{(c_0+h)^2+R_1^2}\right].$$

[Stone.]

12. A spherical cavity of radius b is hollowed out in a sphere of radius a and density ρ, and then completely filled with matter, of density ρ_0. If c is the distance between the centre of the cavity and the centre of the sphere, the attraction due to the composite solid at a point in the line joining these two centres, at a distance d from the centre of the sphere, is

$$\frac{4}{3}\pi\left[\frac{\rho a^3}{d^2}+\frac{b^3(\rho_0-\rho)}{(d\pm c)^2}\right].$$

13. The centre of a sphere of aluminum of radius 10 and of density 2.5, is at the distance 100 from a sphere of the same

size made of gold, of density 19. Show that the attraction due to these spheres is nothing at a point between them, at a distance of about 26.6 from the centre of the aluminum sphere. [Stone.]

14. Show that the attraction at the centre of a sphere of radius r, from which a piece has been cut by a cone of revolution whose vertex is at the centre, is $\pi\rho r \sin^2\alpha$, where α is the half angle of the cone.

15. An iron sphere of radius 10 and density 7 has an eccentric spherical cavity of radius 6, whose centre is at a distance 3 from the centre of the sphere. Find the attraction due to this solid at a point 25 units from the centre of the sphere, and so situated that the line joining it with this centre makes an angle of 45° with the line joining the centre of the sphere and the centre of the cavity. [Stone.]

16. If the piece of a spherical shell of radii r_0 and r_1, intercepted by a cone of revolution whose solid angle is ω and whose vertex is the centre of the shell, be cut out and removed, find the attraction of the remainder of the shell at a point P situated in the axis of the cone at a given distance from the centre of the sphere. If in the vertical shaft of a mine a pendulum be swung, is there any appreciable error in assuming that the only matter whose attraction influences the pendulum lies nearer the centre of the earth, supposed spherical, than the pendulum does?

17. Show that the attraction of a spherical segment is, at its vertex,

$$2\pi h\rho\left\{1-\frac{1}{3}\sqrt{\frac{2h}{a}}\right\},$$

where a is the radius of the sphere and h the height of the segment.

18. Show that the resultant attraction of a spherical segment on a particle at the centre of its base is

$$\frac{2\pi h\rho}{3(a-h)^2}\left[3a^2-3ah+h^2-(2a-h)^{\frac{3}{2}}h^{\frac{1}{2}}\right].$$

19. Show that the attraction at the focus of a segment of a paraboloid of revolution bounded by a plane perpendicular to the axis at a distance b from the vertex is of the form

$$4\pi\rho a \log \frac{a+b}{a}.$$

20. Show that the attraction of the oblate spheroid formed by the revolution of the ellipse of semiaxes a, b, and eccentricity e, is, at the pole of the spheroid,

$$\frac{4\pi\rho b}{e^2}\left\{1-\frac{(1-e^2)^{\frac{1}{2}}}{e}\sin^{-1} e\right\},$$

and that the attraction due to the corresponding prolate spheroid is, at its pole,

$$\frac{4\pi\rho a(1-e^2)}{e^2}\left\{\frac{1}{2e}\log\frac{1+e}{1-e}-1\right\}.$$

21. Show that the attraction at the point $(c, 0, 0)$, due to the homogeneous solid bounded by the planes $x=a$, $x=b$, and by the surface generated by the revolution about the axis of x of the curve $y=f(x)$, is

$$2\pi\rho\int_a^b\left\{1+\frac{c-x}{[(c-x)^2+(fx)^2]^{\frac{1}{2}}}\right\}dx.$$

22. Prove that the attraction of a uniform lamina in the form of a rectangle, at a point P in the straight line drawn through the centre of the lamina at right angles to its plane, is

$$4\mu\sin^{-1}\frac{ab}{\sqrt{a^2+c^2}\sqrt{b^2+c^2}},$$

where $2a$ and $2b$ are the dimensions of the lamina and c the distance of P from its plane.

[Answers to some of these problems and a collection of additional problems illustrative of the text of this chapter may be found near the end of the book.]

CHAPTER II.

THE NEWTONIAN POTENTIAL FUNCTION IN THE CASE OF GRAVITATION.

19. Definition. If we imagine an attracting body M to be cut up into small elements, and add together all the fractions formed by dividing the mass of each element by the distance of one of its points from a given point P in space, the limit of this sum, as the elements are made smaller and smaller, is called the value at P of "the potential function due to M."

If we call this quantity V, we have

$$V = \underset{\Delta m \doteq 0}{\text{limit}} \sum \frac{\Delta m}{r}, \qquad [46]$$

where Δm is the mass of one of the elements and r its distance from P, and where the summation is to include all the elements which go to make up M.

If we denote by $\bar{\rho}$ the average density of the element whose mass is Δm, and call the coördinates of the corner of this element nearest the origin x', y', z', and those of P, x, y, z, we may write

$$\Delta m = \bar{\rho} \Delta x' \Delta y' \Delta z',$$

and

$$V = \iiint \frac{\rho \, dx' dy' dz'}{[(x'-x)^2 + (y'-y)^2 + (z'-z)^2]^{\frac{1}{2}}}, \qquad [47]$$

where ρ is the density at the point (x', y', z'), and where the triple integration is to include the whole of the attracting mass M.

As the position of the point P changes, the value of the quantity under the integral signs in [47] changes, and in general V is a function of the three space coördinates, *i.e.*, $V = f(x, y, z)$.

To avoid circumlocution, a point at which the value of the potential function is V_0 is sometimes said to be "at potential V_0." From the definition of V it is evident that if the value at a point P of the potential function due to a system of masses M_1 existing alone is V_1, and if the value at the same point of the potential function due to another system of masses M_2 existing alone is V_2, the value at P of the potential function due to M_1 and M_2 existing together is $V = V_1 + V_2$.

20. The Derivatives of the Potential Function. If P is a point outside the attracting mass, the quantity

$$\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2},$$

which enters into the expression for V in [47], can never be zero, and the quantity under the integral signs is finite everywhere within the limits of integration; now, since these limits depend only upon the shape and position of the attracting mass and have nothing to do with the coördinates of P, we may differentiate V with respect to either x, y, or z by differentiating under the integral signs. Thus:

$$\begin{aligned} D_xV &= \int\int\int D_x\left[\frac{\rho\, dx'dy'dz'}{r}\right] \\ &= \int\int\int \frac{\rho(x'-x)\,dx'dy'dz'}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{\frac{3}{2}}}, \end{aligned} \qquad [48]$$

where the limits of integration are unchanged by the differentiation. The dexter integral in this equation is (Section 14) the value of the component parallel to the axis of x of the attraction at P due to the given masses, so that we may write, using our old notation,

$$D_xV = X, \qquad [49]$$

and, similarly,

$$D_yV = Y, \qquad [50]$$

$$D_zV = Z. \qquad [51]$$

The resultant attraction at P is

$$R = \sqrt{X^2+Y^2+Z^2} = \sqrt{(D_xV)^2+(D_yV)^2+(D_zV)^2}, \qquad [52]$$

and the direction-cosines of its line of action are:

$$\cos\alpha = \frac{D_x V}{R}, \quad \cos\beta = \frac{D_y V}{R}, \quad \text{and} \quad \cos\gamma = \frac{D_z V}{R}. \qquad [53]$$

It is evident from the definition of the potential function that the value of the latter at any point is independent of the particular system of rectangular axes chosen. If, then, we wish to find the component, in the direction of any line, of the attraction at any point P, we may choose one of our coördinate axes parallel to this line, and, after computing the general value of V, we may differentiate the latter partially with respect to the coördinate measured on the axis in question, and substitute in the result the coördinates of P.

21. Theorem. The results of the last section may be summed up in the words of the following

THEOREM.

*To find the component at a point P, in any direction PK, of the attraction due to any attracting mass M, we may divide the difference between the values of the potential function due to M at P' (a point between P and K on the straight line PK) and at P by the distance PP'. The limit approached by this fraction as P' approaches P is the component required.**

We might have arrived at this theorem in the following way:

If X, Y, Z are the components parallel to the coördinate axes of the attraction at any point P, the component in any direction PK whose direction-cosines are λ, μ, and ν, is

$$\lambda X + \mu Y + \nu Z = \lambda D_x V + \mu D_y V + \nu D_z V. \qquad [54]$$

Let x, y, z be the coördinates of P, and $x + \Delta x$, $y + \Delta y$, $z + \Delta z$ those of P', a neighboring point on the line PK.

---

* If the force is required in absolute kinetic units, the result thus obtained must be multiplied by k, the proper gravitation constant. The reciprocal of k is equal to 1.543×10^7 in the c.g.s. system and to 9.63×10^8 in the f.p.s. system.

If V and V' are the values of the potential function at P and P' respectively, we have, by Taylor's Theorem,

$$V' = V + \Delta x \cdot D_x V + \Delta y \cdot D_y V + \Delta z \cdot D_z V + \epsilon,$$

where ϵ is an infinitesimal of an order higher than the first.

$$\frac{V' - V}{PP'} = \frac{\Delta x}{PP'} \cdot D_x V + \frac{\Delta y}{PP'} \cdot D_y V + \frac{\Delta z}{PP'} \cdot D_z V + \frac{\epsilon}{PP'}; \quad [55]$$

but $\qquad \Delta x = \lambda \cdot PP', \quad \Delta y = \mu \cdot PP', \quad \Delta z = \nu \cdot PP',$

therefore, $\displaystyle \underset{PP' \doteq 0}{\text{limit}} \left(\frac{V' - V}{PP'} \right) = \lambda D_x V + \mu D_y V + \nu D_z V, \quad [56]$

and this (see [54]) is the component in the direction PK of the attraction at P: which was to be proved.

22. The Potential Function everywhere Finite. If P is a point within the attracting mass, the integrand of the expression which gives the value of the potential function at P becomes infinite at P. That V is not infinite in this case is easily proved by making P the origin of a system of polar coördinates as in Section 15, when it will appear that the value of the potential function at P can be expressed in the form

$$V_P = \int \int \int \rho r \sin \theta \, dr \, d\theta \, d\phi; \quad [57]$$

and this is evidently finite.

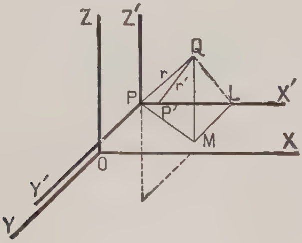

FIG. 15.

Although V_P is everywhere finite, yet when we express its value by means of equation [47], the quantity under the integral signs becomes infinite within the limits of integration, when P is a point inside the attracting mass. Under these circumstances we cannot assume without further proof that the result obtained by differentiating with respect to x under the integral signs is really $D_x V$. It is therefore desirable to compute the limit of the ratio of the difference ($\Delta_x V$) between the values of V at the points $P' = (x + \Delta x, y, z)$ and $P = (x, y, z)$, both within the attracting mass, to the distance (Δx) between these points. For convenience, draw through P (Fig. 15) three lines parallel to the coördinate axes, and let $Q = (x', y', z')$.

Let $PQ = r$, $P'Q = r'$, and $X'PQ = \psi$.

Then

$$r'^2 = r^2 + (\Delta x)^2 - 2r \cdot \Delta x \cdot \cos\psi,$$

where $$\cos\psi = \frac{x' - x}{r},$$

and $$\frac{\Delta_x V}{\Delta x} = \int\int\int\left(\frac{1}{r'} - \frac{1}{r}\right)\frac{\rho\, dx'\, dy'\, dz'}{\Delta x}$$

$$= \int\int\int\left(\frac{r^2 - r'^2}{r' r^2 + r r'^2}\right)\frac{\rho\, dx'\, dy'\, dz'}{\Delta x}$$

$$= \int\int\int\left(\frac{2r\,\Delta x \cos\psi - (\Delta x)^2}{r' r^2 + r r'^2}\right)\frac{\rho\, dx'\, dy'\, dz'}{\Delta x}.$$

Therefore

$$D_x V = \lim_{\Delta x \doteq 0}\left(\frac{\Delta_x V}{\Delta x}\right)$$

$$= \int\int\int \frac{2r\cos\psi}{2r^3} \cdot \rho\, dx'\, dy'\, dz'$$

$$= \int\int\int \frac{\rho\, dx'\, dy'\, dz' \cos\psi}{r^2}. \qquad [58]$$

This last integral is evidently the component parallel to the axis of x of the attraction at P, so that the theorem of Article 21 may be extended to points within the attracting mass.

It is to be noticed that ρ is a function of x', y', and z', but not a function of x, y, and z, and that we have really proved that the derivatives with regard to x, y, and z of

$$\int\int\int \frac{F(x', y', z')}{r}\, dx'\, dy'\, dz',$$

where F is any finite, continuous, and single-valued function of x', y', and z', can always be found by differentiating under the integral signs, whether (x, y, z) is contained within the limits of integration or not.

23. The Potential Function due to a Straight Wire. Let μ be the mass of the unit length of a uniform straight wire AB (Fig. 16) of length $2l$. Take the middle point of the wire for the origin of coördinates, and a line drawn perpendicular to the wire at this point for the axis of x.

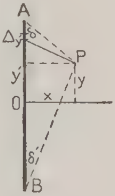

Fig. 16.

The value of the potential function at any point $P(x, y)$ in the coördinate plane is, then, according to [47],

$$V_P = \int_{-l}^{+l} \frac{\mu \, dy'}{[x^2 + (y' - y)^2]^{\frac{1}{2}}} = \mu \Big[\log \{ \sqrt{x^2 + (y' - y)^2} + y' - y \} \Big]_{-l}^{+l}$$

$$= \mu \log \left\{ \frac{l - y + \sqrt{x^2 + (l - y)^2}}{\sqrt{x^2 + (l + y)^2} - l - y} \right\}. \qquad [59]$$

If $r = AP = \sqrt{x^2 + (l - y)^2}$, and $r' = BP = \sqrt{x^2 + (l + y)^2}$, whence $y = \dfrac{r'^2 - r^2}{4l}$, we may eliminate x and y from [59] and express V_P in terms of r and r'.

Thus:

$$V_P = \mu \log \frac{(r + 2l)^2 - r'^2}{r^2 - (r' - 2l)^2} = \mu \log \frac{r + r' + 2l}{r + r' - 2l}. \qquad [60]$$

It is evident from [60] that if P move so as to keep the sum of its distances from the ends of the wire constant, V_P will remain constant. P's locus in this case is an ellipse whose foci are at A and B.

From [59] we get

$$\begin{aligned}
D_x V_P &= \frac{\mu}{x}\left[\frac{x^2}{r[r+(l-y)]} - \frac{x^2}{r'[r'-(l+y)]}\right] \\
&= \frac{\mu}{x}\left[\frac{r^2-(l-y)^2}{r[r+(l-y)]} - \frac{r'^2-(l+y)^2}{r'[r'-(l+y)]}\right] \\
&= \frac{\mu}{x}\left[\frac{r-(l-y)}{r} - \frac{r'+(l+y)}{r'}\right] \\
&= \frac{\mu}{x}\left[1-\cos\delta-1-\cos\delta'\right] \\
&= -\frac{\mu}{x}\left[\cos\delta+\cos\delta'\right],
\end{aligned}$$

and this (Section 6) is the component in the direction of the axis of x of the attraction at P.

24. The Potential Function due to a Spherical Shell. In order to find the value at the point P of the potential function due to a homogeneous spherical shell of density ρ and of radii r_0 and r_1, we may make use of the notation of Section 9.

$$\begin{aligned}
V = \int\!\!\int\!\!\int \frac{\rho r^2 \sin\theta\, dr\, d\theta\, d\phi}{y} &= \int\!\!\int\!\!\int \frac{\rho r\, dy\, dr\, d\phi}{c} \\
&= \frac{2\pi\rho}{c}\int_{r_0}^{r_1} r\, dr \int_{y_0}^{y_1} dy. \qquad [61]
\end{aligned}$$

If P lies within the cavity enclosed by the shell, the limits of y are $(r-c)$ and $(r+c)$, whence

$$V = 2\pi\rho(r_1^2 - r_0^2). \qquad [62]$$

If P lies without the shell, the limits of y are $(c-r)$ and $(c+r)$, whence

$$V = \frac{4}{3}\pi\rho\frac{(r_1^3 - r_0^3)}{c}. \qquad [63]$$

If P is a point within the mass of the shell itself, at a distance c from the centre, we may divide the shell into two parts by means of a spherical surface drawn concentric with the given shell so as to pass through P. The value of the potential function at P is the sum of the components due to these portions of the shell; therefore

$$\begin{aligned} V &= 2\pi\rho(r_1^2 - c^2) + \frac{4}{3}\frac{\pi\rho}{c}(c^3 - r_0^3) \\ &= 2\pi\rho\left\{r_1^2 - \frac{c^2}{3}\right\} - \frac{4\pi\rho}{3c}r_0^3. \qquad [64] \end{aligned}$$

If we put these results together, we shall have the following table: —

	$c < r_0$	$r_0 < c < r_1$	$r_1 < c$
$V =$	$2\pi\rho(r_1^2 - r_0^2)$	$2\pi\rho\left(r_1^2 - \frac{c^2}{3}\right) - \frac{4\pi\rho}{3c}r_0^3$	$\frac{4\pi\rho}{3c}(r_1^3 - r_0^3)$
$D_cV =$	0	$\frac{4\pi\rho}{3}\left(\frac{r_0^3}{c^2} - c\right)$	$-\frac{4\pi\rho}{3c^2}(r_1^3 - r_0^3)$
$D_c^2V =$	0	$-\frac{4\pi\rho}{3}\left(\frac{2r_0^3}{c^3} + 1\right)$	$\frac{8\pi\rho}{3c^3}(r_1^3 - r_0^3)$

If we make V, D_cV, and D_c^2V the ordinates of curves whose abscissas are c, we get Fig. 17.*

Here $LNQS$ represents V, and it is to be noticed that this curve is everywhere finite, continuous, and continuous in direction. The curve $OABC$ represents D_cV. This curve is everywhere finite and continuous, but its direction changes abruptly when the point P enters or leaves the attracting mass. The three disconnected lines OA, DE, and FG represent D_c^2V.

If the density of the shell instead of being uniform were a function of the distance from the centre $[\rho = f(r)]$, we should have at the point P, at the distance c from the centre of the sphere,

$$V = \frac{2\pi}{c}\int_{r_0}^{r_1} f(r)\cdot r\cdot dr \int_{y_0}^{y_1} dy. \qquad [65]$$

---

* See Thomson and Tait's *Treatise on Natural Philosophy*. Notice that $AD = -4\pi\rho$ and that $EF = 4\pi\rho$. For values of c greater than r_1, V, D_cV, and D_c^2V are respectively equal to M/c, $-M/c^2$, and $2M/c^3$, where M is the mass of the shell.

From this it follows, as the reader can easily prove, that the value of the potential function due to a spherical shell whose density is a function of the distance from the centre only is constant throughout the cavity enclosed by the shell, and at all outside points is the same as if the mass of the shell were concentrated at its centre.*

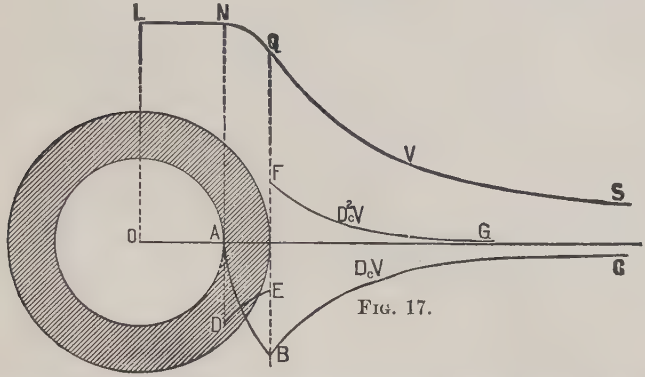

FIG. 17.

25. Equipotential Surfaces. As we have already seen, V is, in general, a function of the three space coördinates [$V = f(x, y, z)$], and in any given case all these points at which the potential function has the particular value c lie on the surface the equation of which is $V = f(x, y, z) = c$.

Such a surface is called an "equipotential" or "level" surface. By giving to c in succession different constant values, the equation $V = c$ yields a whole family of surfaces, and it is always possible to draw through any given point in a field of force a surface at all points of which the potential function has the same value. The potential function cannot have two different values at the same point in space, therefore no two different surfaces of the family $V = c$, where V is the potential function due to an actual distribution of matter, can ever intersect.

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* If the outer radius of the shell be unchanged while the radius of the cavity approaches zero, the values of V and D_cV at O approach as limits the corresponding values at the centre of a solid, homogeneous sphere of density ρ and radius r_1. The value of D_c^2V, however, does not approach as a limit the value of D_c^2V at the centre of such a sphere.

THEOREM.

If there be any resultant force at a point in space, due to any attracting masses, this force acts along the normal to that equipotential surface on which the point lies.

For, let $V = f(x, y, z) = c$ be the equation of the equipotential surface drawn through the point in question, and let the coördinates of this point be x_0, y_0, z_0. The equation of the plane tangent to the surface at the point is

$$(x - x_0) D_{x_0} V + (y - y_0) D_{y_0} V + (z - z_0) D_{z_0} V = 0,$$

and the direction-cosines of any line perpendicular to this plane, and hence of the normal to the given surface at the point (x_0, y_0, z_0), are

$$\cos\alpha = \frac{D_{x_0} V}{\sqrt{(D_{x_0} V)^2 + (D_{y_0} V)^2 + (D_{z_0} V)^2}}, \qquad [66_A]$$

$$\cos\beta = \frac{D_{y_0} V}{\sqrt{(D_{x_0} V)^2 + (D_{y_0} V)^2 + (D_{z_0} V)^2}}, \qquad [66_B]$$

and

$$\cos\gamma = \frac{D_{z_0} V}{\sqrt{(D_{x_0} V)^2 + (D_{\ 0} V)^2 + (D_{z_0} V)^2}}. \qquad [66_C]$$

But if we denote the resultant force of attraction at the point (x_0, y_0, z_0) by R, and its components parallel to the coördinate axes by X, Y, and Z, these cosines are evidently equal to $\frac{X}{R}$, $\frac{Y}{R}$, and $\frac{Z}{R}$ respectively, so that α, β, and γ are the direction-angles not only of the normal to the equipotential surface at the point (x_0, y_0, z_0), but also [35] of the line of action of the resultant force at the point. Hence our theorem.

Fig. 18 represents a meridian section of four of the system of equipotential surfaces due to two equal spheres whose sections are here shaded. The value of the potential function due to two spheres, each of mass M, at a point distant respectively r_1 and r_2 from the centres of the spheres, is

$$V = M\left(\frac{1}{r_1} + \frac{1}{r_2}\right),$$

and if we give to V in this equation different constant values, we shall have the equations of different members of the system of equipotential surfaces. Any one of these surfaces may be easily plotted from its equation by finding corresponding values of r_1 and r_2 which will satisfy the equation; and then, with the centres of the two spheres as centres and these values as radii, describing two spherical surfaces. The intersection of these surfaces, if they intersect at all, will be a line on the surface required.

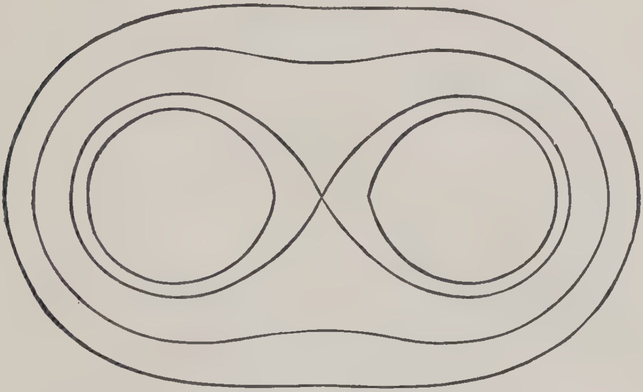

FIG. 18.

If $2a$ is the distance between the centres of the spheres, $V = \frac{2M}{a}$ gives an equipotential surface shaped like an hour-glass. Larger values of V than this give equipotential surfaces, each one of which consists of two separate closed ovals, one surrounding one of the spheres, and the other the other. Values of V less than $\frac{2M}{a}$ give single surfaces which look more and more like ellipsoids the smaller V is.

Several diagrams showing the forms of the equipotential surfaces due to different distributions of matter are given at the end of the first volume of Maxwell's *Treatise on Electricity and Magnetism.*

26. The Value of V at Infinity. The value, at the point P, of the potential function due to any attracting mass M has been defined to be

$$V = \underset{\Delta m \doteq 0}{\text{limit}} \sum \frac{\Delta m}{r}.$$

Let r_0 be the distance of the nearest point of the attracting mass from P, then

$$V < \frac{1}{r_0} \sum \Delta m \text{ or } \frac{M}{r_0}. \qquad [67]$$

The fraction $\frac{M}{r_0}$ has a constant numerator, and a denominator which grows larger without limit the farther P is removed from the attracting masses; hence, we see that, other things being equal, the value at P of the potential function is smaller the farther P is from the attracting matter; and that if P be moved away indefinitely, the value of the potential function at P approaches zero as a limit. In other words, *the value of the potential function at "infinity" is zero.*

About O, any fixed point near the attracting mass, as centre, imagine a spherical surface, S, drawn, of fixed radius, r_0, so large that S shall just include all the distribution. Then, if P is any distant point without S, and if $OP = r$,

$$\frac{M}{r + r_0} < V_P < \frac{M}{r - r_0} \text{ or } \frac{rM}{r + r_0} < rV_P < \frac{rM}{r - r_0}.$$

Since $\underset{r=\infty}{\text{limit}} \frac{r}{r + r_0} = \underset{r=\infty}{\text{limit}} \frac{r}{r - r_0} = 1$, V so vanishes at infinity that the limit of $(r \cdot V_P)$, as r increases without limit, is M.

Since $\frac{M}{(r + r_0)^2} \cos \sin^{-1}\left(\frac{r_0}{r}\right) < -D_r V_P < \frac{M}{(r - r_0)^2}$,

it is easy to see that

$\underset{r=\infty}{\text{limit}} (r^2 D_r V) = -M$ and that $\underset{r=\infty}{\text{limit}} (r^2 D_x V) = -M \cos (x, r)$,

where (x, r) denotes the angle between the axis of x and OP.

27. The Potential Function as a Measure of Work. The amount of work required to move a unit mass, concentrated at a point, from one position, P_1, to another, P_2, by *any* path, in face of the attraction of a system of masses, M, is equal to

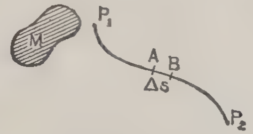

Fig. 19.

$V_1 - V_2$, where V_1 and V_2 are the values at P_1 and P_2 of the potential function due to M.

To prove this, let us divide the given path into equal parts of length Δs, and call the average force which *opposes* the motion of the unit mass on its journey along one of these elements AB (Fig. 19), F. The amount of work required to move the unit mass from A to B is $F\Delta s$, and the whole work done by moving this mass from P_1 to P_2 will be

$$\underset{\Delta s \doteq 0}{\text{limit}} \sum_{P_1}^{P_2} F\Delta s.$$

As Δs is made smaller and smaller, the average force opposing the motion along AB approaches more and more nearly the actual opposing force at A, which is $-D_sV$: therefore

$$\underset{\Delta s \doteq 0}{\text{limit}} \sum_{P_1}^{P_2} F\Delta s = -\int_{P_1}^{P_2} D_s V \cdot ds = V_1 - V_2.$$

It is to be carefully noticed that the *decrease* in the potential function in moving from P_1 to P_2 measures the work required to move the unit mass from P_1 to P_2. If P_2 is removed farther and farther from M, V_2 approaches zero, and $V_1 - V_2$ approaches V_1 as its limit, so that the value at any point P_1, of the potential function, due to any system of attracting masses, is equal to the work which would be required to move a unit mass, supposed concentrated at P_1, from P_1 to "infinity" by any path.

The work (W) that must be done in order to move an attracting mass M' against the attraction of any other mass M, from a given position by any path to "infinity," is the sum of the quantities of work required to move the several elements ($\Delta m'$) into which we may divide M', and this may be written in the form

$$W = \underset{\Delta m' \doteq 0}{\text{limit}} \sum \Delta m' \int\!\!\int\!\!\int \frac{\rho\, dx\, dy\, dz}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{\frac{1}{2}}}$$

$$= \int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int \frac{\rho\rho'\, dx\, dy\, dz\, dx'\, dy'\, dz'}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{\frac{1}{2}}}. \qquad [68]$$

W is called by some writers "the potential of the mass M' with reference to the mass M"; by others, the negative of W is called "the mutual potential energy of M and M'."

In many of the later books on this subject, the word "potential" is never used for the value of the potential function at a point, but is reserved to denote the work required to move a mass from some present position to infinity. If V is the value of the potential function at a point P, at which a mass m is supposed to be concentrated, mV is *the potential of the mass* m. If we could have a unit mass concentrated at a point, *the potential of this mass* and the *value of the potential function at the point* would be numerically identical.

Imagine any given distribution of attracting matter which has the potential function V, divided into elements, of volume $\Delta\tau_1$, $\Delta\tau_2$, $\Delta\tau_3$, $\cdots$, of density ρ_1, ρ_2, ρ_3, $\cdots$, and of mass Δm_1, Δm_2, Δm_3, $\cdots$. If the density at every point in the distribution were λ times what it now is (λ being any positive constant), the potential function would be λV, and, since the volume occupied by each element would be unchanged, the mass of the pth element would be $\lambda\Delta m_p$. To change λ to $\lambda + \Delta\lambda$, the mass of every element must be increased and to the pth element must be brought up the mass-increment $\Delta\lambda \cdot \Delta m_p$. If this quantity were brought up from an infinite distance, the attraction of the existing distribution would do upon it an amount of work represented by $\lambda V \cdot \Delta\lambda \cdot \Delta m_p$, so that the work done on the additions to the whole mass would be $\lambda\,\Delta\lambda\cdot\text{limit}\sum V\Delta m$. The work done by the attractive forces while λ was being changed from λ_0 to λ_1 would be $\text{limit}\sum V\,\Delta m\cdot\int_{\lambda_0}^{\lambda_1}\lambda\,d\lambda$. To find the work done by the attraction for one another of its own parts, while the given distribution is constructed by bringing together its particles from infinite dispersion, we may put $\lambda_0 = 0$, $\lambda_1 = 1$, and get

$$W = \underset{\Delta m \doteq 0}{\text{limit}}\tfrac{1}{2}\sum V\cdot\Delta m,$$

where the summation is to extend over the whole distribution. This quantity, the negative of which (when the matter is attracting) is sometimes called "the intrinsic energy" of the distribution, is given by the formula in attraction units of work. In absolute kinetic work units,

$$W = \tfrac{1}{2}k\int\int\int V\rho\,d\tau.$$

The potential function inside a homogeneous sphere of radius a and density ρ, at a distance r from the centre, being $2\pi\rho\,(a^2 - \frac{1}{3}r^2)$, the intrinsic energy of the sphere is

$$-\int_0^a \pi\rho\,(a^2 - \tfrac{1}{3}r^2)\,4\pi\rho r^2 dr \text{ or } \frac{-16}{15}\pi^2\rho^2a^5 \text{ or } \frac{-3M^2}{5a}$$

attraction units of work. If the c.g.s. system has been used throughout, this is equivalent to $\dfrac{-3M^2}{5a\,(15430000)}$ ergs.

If V and V' are the potential functions due to two neighboring distributions, M and M', if ΔM and $\Delta M'$ are mass elements of the two distributions, and P and P' points in ΔM and $\Delta M'$ respectively, the mutual potential energy of M and M' may be found by integrating $\dfrac{-dM\,dM'}{PP'}$ over both distributions, and, since the order of integration is immaterial, the result may be written $-\int V\,dM'$ or $-\int V'\,dM$.

The intrinsic energy of M and M' considered as a single distribution is to be found by integrating $-\frac{1}{2}(V + V')$ over both masses. This gives $-\tfrac{1}{2}\int V\,dM - \tfrac{1}{2}\int V'\,dM' - \int V\,dM'$ or the sum of the intrinsic energies of M and M' and the mutual energy of the two.

If M and M' were made up of matter every particle of which repelled every other particle according to the Law of Nature, the intrinsic potential energy of M would be $+\tfrac{1}{2}\int V\,dM$ and the mutual potential energy of M and M' would be

$$+\int V\,dM', \text{ or } +\int V'\,dM.$$

28. Laplace's Equation. We have seen that the value of the potential function, and the component in any direction of the attraction at the point P, are always finite functions of the space coördinates, whether P is inside, outside, or at the surface of the attracting masses. We have seen also that by differentiating V at any point in any direction we may find the always finite component in that direction of the attraction at the point. It follows that D_xV, D_yV, D_zV are everywhere finite, and that, in consequence of this, the potential function is everywhere continuous as well as finite.

If P is a point outside of the attracting masses, the quantity under the integral signs in [48], by which $dx'\,dy'\,dz'$ is multiplied, cannot be infinite within the limits of integration, and we can find D_x^2V by differentiating the expression for D_xV under the integral signs.

In this case

$$D_x^2V = \int\int\int \frac{3\,(x'-x)^2 - r^2}{r^5}\,\rho'\,dx'dy'dz', \qquad [69]$$

and similarly,

$$D_y^2V = \int\int\int \frac{3\,(y'-y)^2 - r^2}{r^5}\,\rho'\,dx'dy'dz', \qquad [70]$$

$$D_z^2V = \int\int\int \frac{3\,(z'-z)^2 - r^2}{r^5}\,\rho'\,dx'\,dy'\,dz. \qquad [71]$$

Whence, for all points exterior to the attracting masses,

$$D_x^2 V + D_y^2 V + D_z^2 V = 0. \qquad [72]$$

This is Laplace's Equation. For the operator

$$(D_x^2 + D_y^2 + D_z^2),$$

the symbols δ, Δ, Δ_2, $-\nabla^2$, ∇^2, and $\bar{\nabla}^2$ have been used by different authors, and [72] may be written

$$\nabla^2 V = 0. \qquad [73]$$

The potential function, due to every conceivable distribution of matter, must be such that at all points in empty space Laplace's Equation shall be satisfied.*

29. The Second Derivatives of the Potential Function are Finite at Points within the Attracting Mass. If the point P lies within the attracting mass, V and $D_x V$ are finite, but the quantity under the integral signs in the expression for $D_x V$ becomes infinite within the limits of integration, and we cannot assume that $D_x^2 V$ may be found by differentiating $D_x V$ under the integral signs. In order to find $D_x^2 V$ under these circumstances, it is convenient to transform the equation for $D_x V$. Let us choose our coördinate axes so as to have all the attracting mass in the first octant, and divide the projection of the contour of this mass on the plane yz into elements $(dy'dz')$. Upon each one of these elements let us erect a right prism, cutting the mass twice or some other even number of times. Consider one of the elements $dy'dz'$ the corner of which next the origin has the coördinates 0, y', and z'. The prism erected on this element cuts out elements $ds_1, ds_2, ds_3, ds_4, \cdots ds_{2n}$ from the surface of the attracting mass, and that edge of the prism which is perpendicular to the plane yz at $(0, y', z')$ cuts into the surface at points whose distances from the plane of yz are $a_1, a_3, a_5, \cdots a_{2n-1}$, and out of the surface at points whose distances from the same plane are $a_2, a_4, a_6, \cdots a_{2n}$. At every one of these points of intersection draw a normal towards the interior of the attracting mass, and call the angles which these normals make with the positive direction of the axis of x, α_1, α_2, α_3, $\cdots \alpha_{2n}$. It is to be noticed that α_1, α_3, α_5, $\cdots \alpha_{2n-1}$ are all acute, and that α_2, α_4, α_6, $\cdots \alpha_{2n}$ are all obtuse. The element $dy'dz'$ may be regarded as the common projection of the surface elements ds_1, ds_2, ds_3, $\cdots ds_{2n}$, and, so far as absolute value is concerned, the following equations hold approximately:

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* If a function, continuous with its first derivatives within a region, T, satisfies Laplace's Equation at every point of the region, it is sometimes said to be *harmonic* in T.

$$dy'dz' = ds_1 \cos\alpha_1 = ds_2 \cos\alpha_2 = ds_3 \cos\alpha_3 = \cdots = ds_{2n} \cos\alpha_{2n}.$$

But $dy'dz'$, ds_1, ds_2, ds_3, etc., are all positive areas, and $\cos\alpha_2$, $\cos\alpha_4$, $\cos\alpha_6$, etc., are negative, so that, paying attention to signs as well as to absolute values, we have

$$dy'dz' = +ds_1 \cos\alpha_1 = -ds_2 \cos\alpha_2 = +ds_3 \cos\alpha_3 = -ds_4 \cos\alpha_4 = \text{etc.}$$

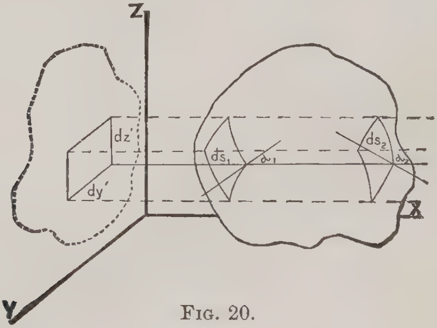

Fig. 20.

Now

$$D_x V = \int\int\int \frac{\rho'(x'-x)dx'dy'dz'}{r^3} = \int\int dy'dz' \int \rho' D_{x}'\left(-\frac{1}{r}\right)dx', \quad [74]$$

and in order to find the value of this expression by the use of the prisms just described, we are to cut each one of these prisms into elementary rectangular parallelopipeds by planes parallel to the plane of yz; we are to multiply the values of every one of these elements which lies within the attracting mass by the value of $\rho' D_x' \left(-\frac{1}{r} \right)$ at its corner next the origin [*i.e.*, at (x', y', z')]; and we are to find the limit of the sum of these as dx' is made smaller and smaller. We are then to compute a like expression for each of the other prisms, and to find the limit of the sum of the whole as the bases of the prisms are made smaller and smaller and their number correspondingly increased.

Wherever the function $\frac{\rho'}{r}$ is a continuous function of x', we have

$$D_x' \frac{\rho'}{r} = \frac{1}{r} D_x' \rho' + \rho' D_x' \frac{1}{r} = \frac{1}{r} D_x' \rho' - \rho' D_x' \left(-\frac{1}{r} \right):$$

hence, if the elementary prisms cut the surface of the attracting mass only twice,

$$D_x V = \int\int dy' dz' \left[-\frac{\rho'}{r} \right]_{x'=a1}^{x=a_2} + \int\int\int \frac{1}{r} D_x' \rho' dx' dy' dz'; \quad [75]$$

and, in general,

$$D_x V = \int\int dy' \, dz' \left[\frac{\rho_1'}{r_1} - \frac{\rho_2'}{r_2} + \frac{\rho_3'}{r_3} - \frac{\rho_4'}{r_4} + \cdots - \frac{\rho_{2n}'}{r_{2n}} \right]$$
$$+ \int\int\int \frac{1}{r} D_x' \rho' \, dx' \, dy' \, dz' \qquad [76]$$

$$= \lim \sum \left(\frac{\rho_1'}{r_1} \cos \alpha_1 ds_1 + \frac{\rho_2'}{r_2} \cos \alpha_2 ds_2 + \frac{\rho_3'}{r_3} \cos \alpha_3 ds_3 + \cdots \right.$$
$$\left. + \frac{\rho_{2n}'}{r_{2n}} \cos \alpha_{2n} ds_{2n} \right) + \int\int\int \frac{1}{r} D_x' \rho' \, dx' \, dy' \, dz', \qquad [77]$$

where $\frac{\rho_k'}{r_k}$ is the value of the quantity $\frac{\rho'}{r}$ at the point where the line $y = y'$, $z = z'$ cuts the surface of the attracting mass for the kth time, counting from the plane yz.

In order to find the value of the limit of the sum which occurs in this expression, it is evident that we may divide the *entire surface* of the attracting mass into elements, multiply the area of each element by the value of $\frac{\rho' \cos \alpha}{r}$ at one of its points, and find the limit of the sum formed by adding all these products together; but this is equivalent to the surface integral of $\frac{\rho' \cos \alpha}{r}$ taken all over the outside of the attracting mass, so that

$$D_x V = \int \frac{\rho'}{r} \cos \alpha \, ds + \int\int\int \frac{D_x' \rho'}{r} \, dx' \, dy' \, dz', \qquad [78]$$

where the first integral is to be taken all over the surface of the attracting mass and the second throughout its volume. This expression for $D_x V$ is in some cases more convenient than that of [48].

We have proved this transformation to be correct, however, only when $\frac{\rho'}{r}$ is finite throughout the attracting mass. If P is a point within the mass, $\frac{\rho'}{r}$ is infinite at P. In this case surround P by a spherical surface of radius ϵ small enough to make the whole sphere enclosed by this surface lie entirely

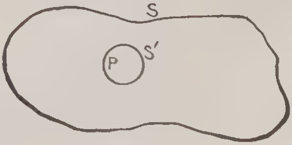

FIG. 21.

within the attracting mass. This is possible unless P lies exactly upon the surface of the attracting mass. Shutting out the little sphere, let V_2 be the potential function due to the rest (T_2) of the attracting mass; then, since P is an outside point, with regard to T_2, we have, by [78],

$$D_x V_2 = \int \frac{\rho'}{\epsilon} \cos \alpha \cdot ds' + \int \frac{\rho'}{r} \cos \alpha \, ds + \int\int\int \frac{D_x' \rho'}{r} \, dx' dy' dz', \; [79]$$

where the first integral is to be extended over the spherical surface, which forms a part of the boundary of the attracting mass to which V_2 is due; the second integral is to be taken over all the rest of the bounding surface of the attracting mass; and the triple integral embraces the volume of all the attracting mass which gives rise to V_2.

As ϵ is made smaller and smaller, V_2 approaches more and more nearly the potential function V, due to all the attracting mass.

In the integral $\int \frac{\rho'}{\epsilon} \cos \alpha \, ds'$, $\cos \alpha$ can never be greater than 1 nor less than -1, so that if $\bar{\rho}'$ is the greatest value of ρ' on the surface of the sphere, the absolute value of the integral must be less than $\frac{\bar{\rho}'}{\epsilon} \int ds'$ or $4\,\pi\bar{\rho}'\epsilon$, and the limit of this as ϵ approaches zero is zero. The second integral in [79] is unaltered by any change in ϵ. If we make P the origin of a system of polar coördinates, it is evident that the triple integral in [79] may be written

$$\int\int\int D_x'\rho' \cdot r \sin \theta \, dr \, d\theta \, d\phi, \qquad [80]$$

and the limit which this approaches as ϵ is made smaller and smaller is evidently finite, for, if $r = 0$, the quantity under the integral sign is zero.

Therefore,

$$\lim_{\epsilon \doteq 0} D_x V_2 = D_x V = \int \frac{\rho'}{r} \cos \alpha \, ds + \int\int\int \frac{D_x'\rho'}{r} \, dx' dy' dz', \quad [81]$$

and [79] is true even when P lies within the attracting mass.

Under the same conditions we have, similarly,

$$D_y V = \int \frac{\rho'}{r} \cos \beta \, ds + \int\int\int \frac{D_y'\rho'}{r} \, dx' \, dy' \, dz, \qquad [82]$$

and

$$D_z V = \int \frac{\rho'}{r} \cos \gamma \, ds + \int\int\int \frac{D_z'\rho'}{r} \, dx' \, dy' \, dz'. \qquad [83]$$

Observing that in these surface integrals r can never be zero, since we have excluded the case where P lies on the surface of the attracting mass, and that the triple integrals belong to the class mentioned in the latter part of Section 22, we will differentiate [81], [82], and [83] with respect to x, y, and z respectively, by differentiating under the integral signs. If the results are finite, we may consider the process allowable.

Performing the work indicated, we have

$$D_x^2 V = \int \rho' \cos\alpha \cdot D_x\left(\frac{1}{r}\right) ds + \int\int\int D_x\left(\frac{1}{r}\right) \cdot D_x'\rho' \cdot dx'dy'dz', \quad [84]$$

$$D_y^2 V = \int \rho' \cos\beta \cdot D_y\left(\frac{1}{r}\right) ds + \int\int\int D_y\left(\frac{1}{r}\right) \cdot D_y'\rho' \cdot dx'dy'dz', \quad [85]$$

$$D_z^2 V = \int \rho' \cos\gamma \cdot D_z\left(\frac{1}{r}\right) ds + \int\int\int D_z\left(\frac{1}{r}\right) \cdot D_z'\rho' \cdot dx'dy'dz', \quad [86]$$

and by making P the centre of a system of polar coördinates and transforming all the triple integrals, it is easy to show that the values of $D_x^2 V$, $D_y^2 V$, $D_z^2 V$ here found are finite, whether P is within or without the attracting mass, if the derivatives of the density are finite. This result* is important.

30. The Derivatives of the Potential Function at the Surface of the Attracting Mass. Let the point P lie on the surface of the attracting mass, or at some other surface where ρ is discontinuous. Make P the centre of a sphere of radius ϵ,

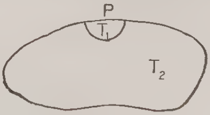

FIG. 22.

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* Lejeune Dirichlet, *Vorlesungen über die im umgekehrten Verhältniss des Quadrats der Entfernung wirkenden Kräfte.*

Riemann, *Schwere, Electricität, und Magnetismus.*

It is to be noticed that while the integral in the second member of [48] represents $D_x V$ even at points within the attracting mass, the integral, I, obtained by differentiating this expression for $D_x V$ under the signs of integration represents $D_x^2 V$ only at outside points. Within the mass I is infinite, while $D_x^2 V$ is finite.

and call the piece which this sphere cuts out of the attracting mass T_1 and the remainder of this mass T_2. Let V_1 and V_2 be the potential functions due respectively to T_1 and T_2, then

$$V = V_1 + V_2, \quad D_x V = D_x V_1 + D_x V_2,$$

and the increment $[\Delta (D_x V)]$ made in $D_x V$ by moving from P to a neighboring point P', inside T_1, is equal to the sum of the corresponding increments $[\Delta (D_x V_1)$ and $\Delta (D_x V_2)]$ made in $D_x V_1$ and $D_x V_2$.

With reference to the space T_2, P is an outside point, so that the values at P of the first derivatives of V_2 with respect to x, y, and z are continuous functions of the space coördinates and $\underset{PP' \doteq 0}{\text{limit}} \Delta (D_x V_2) = 0$.

Let $d\omega$ be the solid angle of an elementary cone whose vertex is at any fixed point O in T_1 used as a centre of coördinates. The element of mass will be $\rho r^2 d\omega\, dr$. The component in the direction of the axis of x of the attraction at O due to T_1 is the limit of the sum taken throughout T_1 of $\dfrac{\rho r^2 a\, d\omega\, dr}{r^2}$, where a is the cosine of the angle which the line joining O with the element in question makes with the axis of x. The difference between the limits of ω is not greater than 4π, and the difference between the limits of r is not greater than 2ϵ. If, then, κ is the greatest value which ρa has in T_1,

$$(D_x V_1)_O < 8\pi\kappa\epsilon.$$

It follows from this that if P' is a point within T_1 so that $PP' < \epsilon$, the change made in $D_x V_1$ by going from P to P' is far less than $16\pi\kappa\epsilon$; but this last quantity can be made as small as we like by making ϵ small enough, so that

$$\underset{PP' \doteq 0}{\text{limit}} \Delta(D_x V_1) = 0,$$

whence

$$\underset{PP' \doteq 0}{\text{limit}} \Delta (D_x V) = \underset{PP' \doteq 0}{\text{limit}} \Delta (D_x V_1) + \underset{PP' \doteq 0}{\text{limit}} \Delta (D_x V_2) = 0,$$

and $D_x V$ varies continuously in passing through P. In a similar manner, it may be proved that $D_y V$ and $D_z V$ are everywhere, even at places where the density is discontinuous, continuous functions of the space coördinates.

The results of the work of the last two sections are well illustrated by Fig. 17. We might prove, with the help of a transformation due to Clausius,* that the second derivatives of the potential function are finite at all points on the surface of the attracting matter where the curvature is finite, but that the normal second derivatives generally change their values abruptly whenever the point P crosses a surface at which ρ is discontinuous, as at the surface of the attracting masses. The fact, however, that this last is true in the special case of a homogeneous spherical shell suffices to show that we cannot expect all the second derivatives of V to have definite values at the boundaries of attracting bodies.

31. Gauss's Theorem. If any closed surface S drawn in a field of force be divided up into a large number of surface elements, and if each one of these elements be multiplied by the component, in the direction of the interior normal of the force of attraction at a point of the element, and if these products be added together, the limit of the sum thus obtained is called the "surface integral of normal attraction over S."

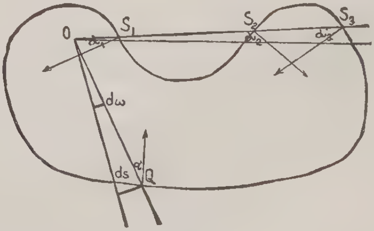

FIG. 23.

If any closed surface S be described so as to shut in completely a mass m concentrated at a point, the surface integral of normal attraction due to m, taken over S, is $4\pi m$; and, in general, if any closed surface S be described so as to shut in completely any system of attracting masses M, the surface integral over S of the normal attraction due to M is $4\pi M$.

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* *Die Potentialfunction und das Potential*, §§ 19–24.

In order to prove this, divide S up into surface elements, and consider one of these ds at Q. The attraction at Q in the direction QO, due to the mass m concentrated at O, is $\frac{m}{\overline{QO}^2} = \frac{m}{r^2}$. The component of this in the direction of the interior normal is $\frac{m}{r^2}\cos\alpha$, and the contribution which ds yields to the sum whose limit is the surface integral required is $\frac{m\cos\alpha\, ds}{r^2}$. Connect every point of the perimeter of ds with O by a straight line, thus forming a cone of such size as to cut out of a spherical surface of unit radius drawn about O an element $d\omega$, say. If we draw about O a sphere of radius $r = OQ$, the cone will intercept on its surface an element equal to $r^2 \cdot d\omega$. This element is the projection on the spherical surface of ds; hence $ds \cos\alpha = r^2 d\omega$, approximately, and the contribution of the element ds to our surface integral is $m\, d\omega$. But an elementary cone may cut the surface more than once; indeed, any odd number of times. Consider such a cone, one element of which cuts the surface thrice in S_1, S_2, and S_3. Let OS_1, OS_2, and OS_3 be called r_1, r_2, and r_3 respectively, and let the surface elements cut out of S by the cone be ds_1, ds_2, and ds_3, and the angles between the line S_3O and the interior normals to S at S_1, S_2, and S_3 be α_1, α_2, α_3. It is to be noticed that when the cone cuts out of S, the corresponding angle is acute, and that when it cuts in, the corresponding angle is obtuse. α_1 and α_3 are acute, and α_2 obtuse. If we draw about O three spherical surfaces with radii r_1, r_2, and r_3 respectively, the cone will cut out of these the elements $r_1^2 d\omega$, $r_2^2 d\omega$, and $r_3^2 d\omega$. In absolute size, $ds_1 = r_1^2 d\omega \sec\alpha_1$, $ds_2 = r_2^2 d\omega \sec\alpha_2$, and $ds_3 = r_3^2 d\omega \sec\alpha_3$, approximately, but ds_2 and $r^2 d\omega$ are both positive, being areas, and $\sec\alpha_2$ is negative. Taking account of sign, then,

$ds_2 = -r^2 d\omega \sec \alpha_2$, and the cone's three elements yield to the surface integral of normal attraction the quantity

$$m\left(\frac{ds_1 \cos \alpha_1}{r_1^{\,2}} + \frac{ds_2 \cos \alpha_2}{r_2^{\,2}} + \frac{ds_3 \cos \alpha_3}{r_3^{\,2}}\right) = m\,(d\omega - d\omega + d\omega) = m\,d\omega.$$

However many times the cone cuts S, it will yield $m\,d\omega$ to the surface integral required: all such elementary cones will yield then $m\sum d\omega = m\,4\,\pi$, if S is closed, and, in general, $m\Theta$, where Θ is the solid angle which S subtends at O.

If, instead of a mass concentrated at a point, we have any distribution of masses, we may divide these into elements, and apply to each element the theorem just proved; hence our general statement.

If from a point O without a closed surface S an elementary cone be drawn, the cone, if it cuts S at all, will cut it an even number of times. Using the notation just explained, the contribution which any such cone will yield to the surface integral taken over S of a mass m concentrated at O is

$$m\left(\frac{ds_1 \cos \alpha_1}{r_1^{\,2}} + \frac{ds_2 \cos \alpha_2}{r_2^{\,2}} + \frac{ds_3 \cos \alpha_3}{r_3^{\,2}} + \frac{ds_4 \cos \alpha_4}{r_4^{\,2}} + \cdots\right)$$
$$= m\,(-\,d\omega + d\omega - d\omega + d\omega - \cdots) = m \cdot 0 = 0,$$

and the surface integral over any closed surface of the normal attraction due to any system of outside masses is zero.

The results proved above may be put together and stated in the form of a

THEOREM DUE TO GAUSS.

If there be any distribution of matter partly within and partly without a closed surface S, and if M be the sum of the masses which S encloses, and M' the sum of the masses outside S, the surface integral over S of the normal attraction N toward the interior, due to both M and M', is equal to $4\,\pi M$. If V be the potential function due to both M and M', we have

$$\int N\,ds = \int D_n V \cdot ds = 4\,\pi M.$$

It is easy to see that if a mass M be supposed concentrated *on* any closed surface S the curvature of which is everywhere finite, the surface integral of normal attraction taken over S will be $2\pi M$; for all the elementary cones which can be drawn from a point P on the surface so as to cut S once or some other odd number of times, lie on one side of the tangent plane at the point, and intercept just half the surface of the sphere of unit radius the centre of which is P.

From Gauss's Theorem it follows immediately that at some parts of a closed surface situated in a field of force, but enclosing none of the attracting mass, the normal component of the resultant attraction must act towards the interior of the surface and at some parts toward the exterior, for otherwise the limit of the sum of the intrinsically positive elements of the surface, each one multiplied by the component in the direction of the interior normal of the attraction at one of its own points, could not be zero. In other words, the potential function, the rate of change of which measures the attraction, must at some parts of the surface increase and at others decrease in the direction of the interior normal.

32. Tubes of Force. A line which cuts orthogonally the different members of the system of equipotential surfaces corresponding to any distribution of matter is called a "line of force," since its direction at each point of its course shows the direction of the resultant force at the point. If through all points of the contour of a portion of an equipotential surface lines of force be drawn, these lines lie on a surface called a "tube of force." We may easily apply Gauss's Theorem to a space cut out and bounded by a portion of a tube of force and two equipotential surfaces; for the sides of the tube do not con-

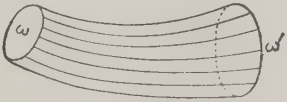

FIG. 24.

tribute anything to the surface integral of normal attraction, and the resultant force is all normal at points in the equipotential surfaces. If ω and ω' are the areas of the sections of a tube of force made by two equipotential surfaces, and if F and F' are the average interior forces on ω and ω', we have

$$F\omega + F'\omega' = 0 \qquad [87]$$

if the tube encloses empty space, and

$$F\omega + F'\omega' = 4\pi m \qquad [88]$$

when the tube encloses a mass m of attracting matter.

33. Spherical Distributions. In the case of a distribution about a point in spherical shells, so that the density is a function of the distance from this point only, the lines of force are straight lines whose directions all pass through the central point. Every tube of force is conical, and the areas cut out of different equipotential surfaces by a given tube of force are proportional to the square of the distance from the centre.

Consider a tube of force which intercepts an area ψ from a spherical surface of unit radius drawn with O as a centre, and apply Gauss's Theorem to a box cut out of this tube by two equipotential surfaces of radii r and $(r + \Delta r)$ respectively.

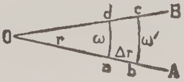

FIG. 25.

Let AOB (Fig. 25) be a section of the tube in question. The area of the portion of spherical surface ω which is represented in section at ad is $r^2\psi$, and the area of that at bc is $(r + \Delta r)^2\psi$. If the average force acting on ω toward the inside of the box is F, the average force acting on ω' toward the inside of the box will be $-(F + \Delta_r F)$, and the surface integral of normal attraction taken all over the outside of the box is

$$Fr^2\psi - (F + \Delta_r F)(r + \Delta r)^2\psi = -\psi \cdot \Delta_r(F \cdot r^2) \qquad [89]$$

If the tube of force which we have been considering be extended far enough, it will cut all the concentric layers of matter, traverse all the empty regions between the layers, if there are such, and finally emerge into outside space.

If we choose r so that the box shall contain no matter, the surface integral taken over the box must be zero.

In this case,

$$-\psi\Delta_r(Fr^2)=0,$$

therefore,

$$F=\frac{c}{r^2}, \qquad [90]$$

and

$$V=-\frac{c}{r}+\mu. \qquad [91]$$

From this it follows that in a region of empty space, either included between the two members of a system of concentric spherical shells of density depending only upon the distance from the centre, or outside the whole system, the force of attraction at different points varies inversely as the squares of the distances of these points from the centre.

Suppose that the box $(abcd)$ lies in a shell whose density is constant; then the surface integral of normal attraction taken over the box is equal to 4π times the matter within the box. In this case the quantity of matter inside the box is

$$\rho\tfrac{4}{3}\pi[(r+\Delta r)^3-r^3]\frac{\psi}{4\pi} \text{ or } \rho\psi r^2\Delta r+\epsilon,$$

where ϵ is an infinitesimal of an order higher than the first. Therefore,

$$-\psi\Delta_r(Fr^2)=4\pi(\rho\psi r^2\Delta r+\epsilon),$$

or

$$\underset{\Delta r \doteq 0}{\text{limit}}\frac{\Delta_r(Fr^2)}{\Delta r}=-4\pi\rho r^2,$$

whence

$$F=-\frac{4\pi\rho r}{3}+\frac{c}{r^2}, \qquad [92]$$

and

$$V=-\frac{c}{r}-\frac{2}{3}\pi\rho r^2+\mu. \qquad [93]$$

If the box lies in a shell whose density is inversely proportional to the distance from the centre, we shall have

$$\underset{\Delta r \doteq 0}{\text{limit}} \frac{\Delta_r (Fr^2)}{\Delta r} = -4\pi \left(\frac{\lambda}{r}\right) r^2, \qquad [94]$$

whence

$$F = -2\pi\lambda + \frac{c}{r^2}, \qquad [95]$$

and

$$V = -\frac{c}{r} - 2\pi\lambda r + \mu. \qquad [96]$$

In general, if the box lies in a shell whose density is $f(r)$, we shall have

$$\underset{\Delta r \doteq 0}{\text{limit}} \frac{\Delta_r (Fr^2)}{\Delta r} = -4\pi f(r) r^2, \qquad [97]$$

whence

$$F = \frac{c}{r^2} - \frac{4\pi}{r^2} \int f(r) r^2 \cdot dr. \qquad [98]$$

In order to learn how to use the results just obtained to determine the force of attraction at any point due to a given spherical distribution, let us consider the simple case of a single shell, of radii 4 and 5, and of density [λr] proportional to the distance from the centre.

At points within the cavity enclosed by the shell we must have, according to [90] and [91],

$$F = \frac{c}{r^2} \quad \text{and} \quad V = -\frac{c}{r} + \mu;$$

But the force is evidently zero at the centre of the shell, where r is zero, so that c must be zero everywhere within the cavity, and there is no resultant force at any point in the region. The value, at the centre, of the potential function due to the shell is evidently

$$\mu = \int_4^5 4\pi\lambda r^2 dr = \frac{244\pi\lambda}{3}, \qquad [99]$$

and it has the same value at all other points in the cavity.

In the shell itself it is easy to see that we must have at all points,

$$F = \frac{c'}{r^2} - \pi\lambda r^2 \quad \text{and} \quad V = -\frac{c'}{r} - \frac{\pi\lambda r^3}{3} + \mu'. \qquad [100]$$

In order to determine the constants in this equation, we may make use of the fact that F and V are everywhere continuous functions of the space coördinates, so that the values of F and V obtained by putting $r = 4$, the inner radius of the shell, in [100], must be the same as those obtained by making $r = 4$ in the expressions which give the values of F and V for the cavity enclosed by the shell. This gives us

$$c' = 256\,\pi\lambda \quad \text{and} \quad \mu' = \frac{500\,\pi\lambda}{3},$$

so that for points within the mass of the shell we have

$$F = \frac{256\,\pi\lambda}{r^2} - \pi\lambda r^2, \qquad [101]$$

and

$$V = -\frac{256\,\pi\lambda}{r} - \frac{\pi\lambda r^3}{3} + \frac{500\,\pi\lambda}{3}. \qquad [102]$$

For points without the shell we have the same general expressions for F and V as for points within the cavity enclosed by the shell, namely,

$$F = \frac{k}{r^2} \quad \text{and} \quad V = -\frac{k}{r} + m, \qquad [103]$$

but the constants are different for the two regions.

Keeping in mind the fact that F and V are continuous, it is easy to see that we must get the same result at the boundary of the shell, where $r = 5$, whether we use [103], or [101] and [102].

This gives

$$k = -369\,\pi\lambda \quad \text{and} \quad m = 0;$$

so that for all points outside the shell we have

$$F = -\frac{369\,\pi\lambda}{r^2}, \qquad [104]$$

and

$$V = \frac{369\,\pi\lambda}{r}. \qquad [105]$$

These last results agree with the statements made in Section 13, for the mass of the shell is $369\,\pi\lambda$.

The values, at every point in space, of the potential function and of the attraction due to any spherical distribution may be found by determining, first, with the aid of Gauss's Theorem, the general expressions for F and V in the several regions; then the constants for the innermost region, remembering that there is no resultant attraction at the centre of the system; and finally, in succession (moving from within outwards), the constants for the other regions, from a consideration of the fact that no abrupt change in the values of either F or V is made by crossing the common boundary of two regions.

This method of treating problems is of great practical importance.

34. Cylindrical Distributions. In the case of a cylindrical distribution about an axis, where the density is a function of the distance from the axis only, the equipotential surfaces are concentric cylinders of revolution; the lines of force are straight lines perpendicular to the axis; and every tube of force is a wedge.

If we apply Gauss's Theorem to a box shut in between two equipotential surfaces of radii r and $r + \Delta r$, two planes perpendicular to the axis, and two planes passing through the axis,

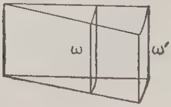

Fig. 26.

we have, if ψ is the area of the piece cut out of the cylindrical surface of unit radius by our tube of force,

$$\omega = r \cdot \psi, \quad \omega' = (r + \Delta r) \cdot \psi,$$

and for the surface integral of normal attraction taken over the box,

$$F\omega + F'\omega' = -\psi \cdot \Delta_r (r \cdot F). \qquad [106]$$

If our box is in empty space,

$$\Delta_r (r \cdot F) = 0,$$

so that

$$F = \frac{c}{r} \quad \text{and} \quad V = c \log r + \mu. \qquad [107]$$

If the box is within a shell of constant density ρ,

$$-\psi \cdot \Delta_r(r \cdot F) = 4\pi\psi\,\rho r\,\Delta r,$$

so that $$F = \frac{c}{r} - 2\pi\rho r \quad \text{and} \quad V = c\log r - \pi\rho r^2 + \mu. \qquad [108]$$

35. Poisson's Equation. Let us now apply Gauss's Theorem to the case where our closed surface is that of an element of volume of an attracting mass in which ρ is either constant or a continuous function of the space coördinates. We will consider three cases, using first rectangular coördinates, then cylinder coördinates, and finally spherical coördinates.

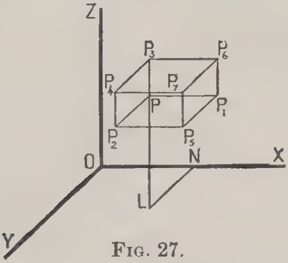

Fig. 27.

I. In the first case, our element is a rectangular parallelopiped (Fig. 27). Perpendicular to the axis of x are two equal surfaces of area $\Delta y \cdot \Delta z$, one at a distance x from the plane yz, and one at a distance $x + \Delta x$ from the same plane. The average force perpendicular to a plane area of size $\Delta y \Delta z$, parallel to the plane yz, and with edges parallel to the axes of y and z, is evidently some function of the coördinates of the corner of the element nearest the origin.

That is, if $P = (x, y, z)$, the average force on PP_4 parallel to the axis of x is $X = f(x, y, z)$, and the average force on P_1P_7 in the same direction is $f(x + \Delta x, y, z) = X + \Delta_x X$, so that PP_4 and P_1P_7 yield towards the surface integral of interior-normal attraction taken over the element, the quantity $-\Delta x \Delta y \Delta z \cdot \dfrac{\Delta_x X}{\Delta x}$. Similarly, the other two pairs of elementary surfaces yield

$-\Delta x \Delta y \Delta z \dfrac{\Delta_y Y}{\Delta y}$ and $-\Delta x \Delta y \Delta z \dfrac{\Delta_z Z}{\Delta z}$, and, if ρ_0 is the average density of the matter enclosed by the box, we have

$$-\Delta x \Delta y \Delta z \left[\frac{\Delta_x X}{\Delta x} + \frac{\Delta_y Y}{\Delta y} + \frac{\Delta_z Z}{\Delta z}\right] = 4\pi\rho_0 \Delta x \Delta y \Delta z. \quad [109]$$

This equation is true whatever the size of the element $\Delta x \Delta y \Delta z$. If this element is made smaller and smaller, the average normal force $[X]$ on PP_4 approaches in value the force $[D_x V]$ at P in the direction of the axis of x; Y and Z approach respectively the limits $D_y V$ and $D_z V$; and ρ_0 approaches as its limit the actual density $[\rho]$ at P.

Taking the limits of both sides of [109], after dividing by $\Delta x \Delta y \Delta z$, we have

$$D_x^2 V + D_y^2 V + D_z^2 V = -4\pi\rho,$$

or

$$\nabla^2 V = -4\pi\rho, \quad [110]$$

which is Poisson's Equation. The potential function due to any conceivable distribution of attracting matter must be such that at all points within the attracting mass this equation shall be satisfied.

For points in empty space $\rho = 0$, and Poisson's Equation degenerates to Laplace's Equation.

II. In the case of cylindrical coördinates, the element of volume (Fig. 28) is bounded by two cylindrical surfaces of revolution having the axis of z as their common axis and radii r and $r + \Delta r$, two planes perpendicular to this axis and distant Δz

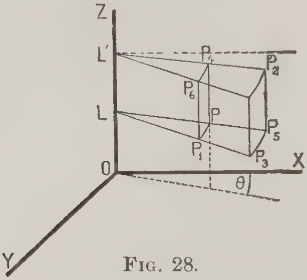

Fig. 28.

from each other, and two planes passing through the axis and forming with each other the diedral angle $\Delta\theta$.

Call R, Θ, and Z the average normal forces upon the elementary planes PP_6, PP_2, and PP_3 respectively, then the surface integral of normal attraction over the volume element will be

$$-\Delta\theta\,\Delta z\,\Delta_r(r\cdot R)-\Delta r\,\Delta z\,\Delta_\theta\Theta-\Delta\theta\left[r\Delta r+\tfrac{1}{2}(\Delta r)^2\right]\Delta_z Z$$
$$=4\,\pi\rho_0\ (\text{vol. of box})\,; \qquad [111]$$

whence, approximately,

$$\frac{1}{r}\frac{\Delta_r(rR)}{\Delta r}+\frac{1}{r}\frac{\Delta_\theta\Theta}{\Delta\theta}+\frac{\Delta_z Z}{\Delta z}=-\,4\,\pi\rho_0\,\frac{\text{vol. of box}}{r\,\Delta r\,\Delta\theta\,\Delta z}. \qquad [112]$$

The force at P in direction PP_5 is D_rV, in direction PP_4 is D_zV, and perpendicular to LP in the plane PLP_1 is $\frac{1}{r}\cdot D_\theta V$, so that if the box is made smaller and smaller, our equation approaches the form

$$\frac{1}{r}D_r(r\cdot D_rV)+\frac{1}{r^2}D_\theta^{\,2}V+D_z^{\,2}V=-\,4\,\pi\rho. \qquad [113]$$

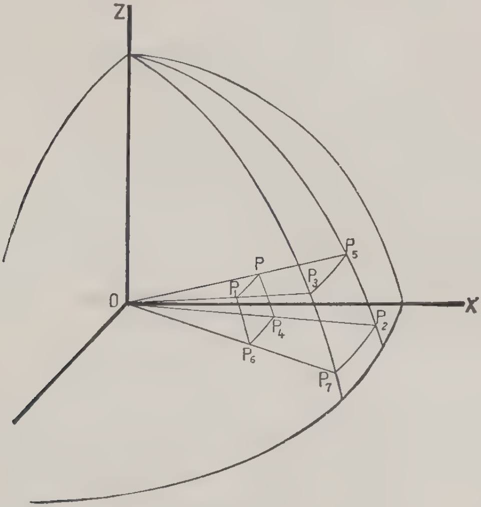

Fig. 29.

III. In the case of spherical coördinates, the volume element is of the shape shown in Fig. 29. Let $OP=r$, $ZOP=\theta$, and denote by ϕ the diedral angle between the planes ZOP and ZOX. Denote by R, Θ, and Φ the average normal forces on the faces PP_6, PP_3, and PP_2 respectively; then the surface integral of normal attraction over the elementary box is approximately

$$-\sin\theta\Delta\theta\Delta\phi\cdot\Delta_r(r^2R)-r\Delta\theta\Delta r\Delta_\phi\Phi-r\Delta\phi\Delta r\cdot\Delta_\theta(\sin\theta\cdot\Theta)$$
$$=4\pi\rho_0\cdot(\text{vol. of box}); \qquad [114]$$

whence
$$\frac{1}{r^2}\cdot\frac{\Delta_r(r^2R)}{\Delta r}+\frac{1}{r\sin\theta}\cdot\frac{\Delta_\phi\Phi}{\Delta\phi}+\frac{1}{r\sin\theta}\cdot\frac{\Delta_\theta(\sin\theta\cdot\Theta)}{\Delta\theta}$$
$$=-4\pi\rho_0\cdot\frac{\text{vol. of box}}{r^2\sin\theta\Delta r\Delta\theta\Delta\phi}. \qquad [115]$$

The force at P in the direction PP_5 is D_rV, in the direction PP_1 is $\frac{1}{r\sin\theta}\cdot D_\phi V$, and in the direction PP_4 is $\frac{1}{r}\cdot D_\theta V$; therefore, as the element of volume is made smaller and smaller, our equation approaches the form

$$\sin\theta\cdot D_r(r^2D_rV)+\frac{D_\phi{}^2V}{\sin\theta}+D_\theta(\sin\theta\cdot D_\theta V)$$
$$=-4\pi\rho r^2\sin\theta. \qquad [116]$$

This equation, as well as that for cylinder coördinates, might have been obtained by transformation from the equation in rectangular coördinates.

We may devote the rest of this section to the stating of some general results which will be intelligible only to those readers who are familiar with the theory and the use of curvilinear coördinates.

If u, v, w are any three analytic functions of x, y, z which define a set of orthogonal curvilinear coördinates, and if $h_u{}^2=(D_xu)^2+(D_yu)^2+(D_zu)^2$, $h_v{}^2=(D_xv)^2+(D_yv)^2+(D_zv)^2$, $h_w{}^2=(D_xw)^2+(D_yw)^2+(D_zw)^2$, it is possible to show that Poisson's Equation may be written in either of the forms

$$D_u{}^2V\cdot h_u{}^2+D_v{}^2V\cdot h_v{}^2+D_w{}^2V\cdot h_w{}^2+D_uV\cdot\nabla^2u$$
$$+D_vV\cdot\nabla^2v+D_wV\cdot\nabla^2w=-4\pi\rho,$$

$$h_u \cdot h_v \cdot h_w \left\{ D_u\left(\frac{h_u \cdot D_u V}{h_v \cdot h_w}\right) + D_v\left(\frac{h_v \cdot D_v V}{h_u \cdot h_w}\right) \right.$$
$$\left. + D_w\left(\frac{h_w \cdot D_w V}{h_u \cdot h_v}\right) \right\} = -4\pi\rho.$$

By giving to c in the equation $u = c$, where u is a given function of (x, y, z), different values in succession we may get the equations of any number of surfaces on each of which u is constant. These surfaces may or may not be the equipotential surfaces of a possible distribution of matter. If they are, it must be possible to find a potential function which changes only when u changes and is, therefore, a function of u only. We may in this case consider u as one of a set of three orthogonal curvilinear coördinates (u, v, w), and since, by hypothesis, $D_v V = 0$, and $D_w V = 0$, we may write Laplace's Equation in the form $D_u^2 V \cdot h_u^2 + D_u V \cdot \nabla^2 u = 0$, or

$$D_u^2 V + D_u V\left(\frac{\nabla^2 u}{h_u^2}\right) = 0.$$

If now the ratio of $\nabla^2 u$ to h_u^2 is expressible as a function of u only, the equation is an ordinary differential equation the solution of which gives the most general solution of Laplace's Equation which is a function of u only. If, however, the ratio of $\nabla^2 u$ to h_u^2 is not expressible as a function of u only, V, which by hypothesis involves u only, must satisfy a differential equation which involves besides u one or both of the other coördinates v and w, so that we infer that no solution of Laplace's Equation exists which is everywhere a function of u only. A set of confocal ellipsoidal surfaces forms a possible set of equipotential surfaces, while a family of concentric, similar, and similarly placed ellipsoidal surfaces cannot be the level surfaces in empty space of any distribution of matter. Two concentric, similar, and similarly placed ellipsoidal surfaces, S_1 and S_2, may be equipotential but, in this case, the level surfaces between S_1 and S_2 will not be ellipsoidal surfaces similar to them.

36. Poisson's Equation in the Integral Form. In [109] X may be regarded as a function of x, y, z, Δy, and Δz, which approaches $D_x V$ as a limit when Δy and Δz are made to approach zero, and it may not be evident that the limit, when Δx, Δy, and Δz are together made to approach zero, of the fraction $\frac{\Delta_x X}{\Delta x}$ is $D_x^2 V$. For this reason it is worth while to establish Poisson's Equation by another method.

It is shown in Section 29 that the volume integral of the quantity $-D_x\left(\frac{\rho}{r}\right)$, taken throughout a certain region, is the surface integral of $\frac{\rho}{r}\cos\alpha$ taken all over the surface which bounds the region. In this proof we might substitute for $\frac{\rho}{r}$ any other function of the three space coördinates which throughout the region is finite, continuous, and single-valued, and state the results in the shape of the following theorem:

If T is any closed surface and U a function of x, y, and z which for every point inside T has a finite, definite value which changes continuously in moving to a neighboring point, then

$$\int\!\!\int\!\!\int D_x U \cdot dx\,dy\,dz = -\int U\cos\alpha\,ds, \qquad [117]$$

$$\int\!\!\int\!\!\int D_y U \cdot dx\,dy\,dz = -\int U\cos\beta\,ds, \qquad [118]$$

and

$$\int\!\!\int\!\!\int D_z U \cdot dx\,dy\,dz = -\int U\cos\gamma\,ds, \qquad [119]$$

where α, β, and γ are the angles made by the interior normals at the various points of the surface with the positive direction of the coördinate axes, and where the sinister integrals are to be extended all through the space enclosed by T, and the dexter integrals all over the bounding surface.

If we apply this theorem to an imaginary closed surface which shuts in any attracting mass of density either uniform or variable, and if for U in [117], [118], and [119] we use respectively

$D_x V$, $D_y V$, and $D_z V$, and add the resulting equations together, we shall have

$$\iiint (D_x^2 V + D_y^2 V + D_z^2 V)\, dx\, dy\, dz$$
$$= -\int (D_x V \cos\alpha + D_y V \cos\beta + D_z V \cos\gamma)\, ds. \quad [120]$$

The integral in the second member of this equation is evidently (see [56]) the surface integral of normal attraction taken over our imaginary closed surface, and this by Gauss's Theorem is equal to 4π times the quantity of matter inside the surface, so that

$$\iiint (D_x^2 V + D_y^2 V + D_z^2 V)\, dx\, dy\, dz$$
$$= -4\pi \iiint \rho\, dx\, dy\, dz. \quad [121]$$

Since this equation is true whatever the form of the closed surface, we must have at every point

$$D_x^2 V + D_y^2 V + D_z^2 V = -4\pi\rho.$$

For if throughout any region $\nabla^2 V$ were greater than $-4\pi\rho$, we might take the boundary of this region as our imaginary surface. In this case every term in the sum whose limit gives the sinister of [121] would be greater than the corresponding term in the dexter, so that the equation would not be true. Similar reasoning shuts out the possibility of $\nabla^2 V$'s being less than $-4\pi\rho$.

37. The Average Value of the Potential Function on a Spherical Surface. If, in a field of force due to a mass m concentrated at a point P, we imagine a spherical surface to be drawn so as to exclude P, the surface integral taken over this surface of the value of the potential function due to m is equal to the area of the surface multiplied by the value of the potential function at the centre of the sphere.

To prove this, let the radius of the sphere be a and the distance $[OP]$ of P from its centre c. Take the centre of the sphere for origin and the line OP for the axis of x. Divide the surface of the sphere into zones by means of a series of planes cutting the axis of x perpendicularly at intervals of Δx. The area of each one of these zones is $2\pi a\,dx$, so that the surface integral of $\frac{m}{r}$ is

$$\int_{-a}^{+a}\frac{m\,2\pi a\,dx}{\sqrt{a^2+c^2-2cx}}=-\left[\frac{2\pi ma\sqrt{a^2+c^2-2cx}}{c}\right]_{-a}^{+a},$$

and the value of this, since the radical represents a positive quantity, is $\frac{4\pi a^2 m}{c}$, which proves the proposition.

The surface integral of the potential function taken over the sphere, divided by the area of the sphere, is often called "the average value of the potential function on the spherical surface."

If we have any distribution of attracting matter, we may divide it into elements, apply the theorem just proved to each of these elements, and, since the potential function due to the whole distribution is the sum of those due to its parts, assert that:

The average value on a spherical surface of the potential function due to any distribution of matter entirely outside the sphere is equal to the value of the potential function at the centre of the sphere.

If a function, U, of the space coördinates attains a maximum (or a minimum) value at a point, Q, it is possible to draw about Q as centre a spherical surface, S, of radius so small that the value of U at every point of S shall be less (or greater) than the value of U at Q. It follows, therefore, from the theorem just stated that:

The potential function due to a finite distribution of matter cannot attain either a maximum or a minimum value at any point in empty space.

We may infer from the first of the theorems just stated that, if the potential function is constant within any closed surface, S, drawn in a region, T, which contains no matter, it will have the same value in those parts of T which lie outside S. For, if the values of the potential function at points in empty space just outside S were different from the value inside, it would always be possible to draw a sphere of which the centre should be inside S, and which outside S should include only such points as were all at either higher or lower potential than the space inside S; but in this case the value of the potential function at the centre of the sphere would not be the average of its values over its surface. A more satisfactory proof can be given with the help of Spherical Harmonics.

The value of the potential function cannot be constant in unlimited empty space surrounding an attracting mass M, for, if it were, we could surround the mass by a surface over which the surface integral of normal attraction would be zero instead of $4\pi M$.

The average value on a spherical surface of the potential function [V], due to any distribution [M] of attracting matter wholly within the surface, is the same as if M were concentrated at the centre O of the space which the surface encloses. For the average values [V_0 and $V_0 + \Delta_r V_0$] of V on concentric spherical surfaces of radii r and $r + \Delta r$ may be written $\frac{1}{4\pi r^2}\int V ds$ (or $\frac{1}{4\pi}\int V d\omega$, if $d\omega$ is the solid angle of an elementary cone with vertex at O, which intercepts the element ds from the surface of a sphere of radius r), and $\frac{1}{4\pi}\int (V + \Delta_r V) d\omega$;

whence
$$\Delta_r V_0 = \frac{1}{4\pi}\int \Delta_r V \cdot d\omega,$$

and
$$D_r V_0 = \frac{1}{4\pi}\int D_r V \cdot d\omega.$$

Now $-\int D_r V \cdot a^2 d\omega$ is the integral of normal attraction taken over the spherical surface, whence, by Gauss's Theorem,

$$D_r V_0 = -\frac{4\pi M}{4\pi r^2}, \quad \text{and} \quad V_0 = \frac{M}{r} + 0,$$

since $V_0 = 0$, for $r = \infty$.

38. The Equilibrium of Fluids at Rest under the Action of Given Forces. Elementary principles of Hydrostatics teach us that when an incompressible fluid is at rest under the action of any system of applied forces, the hydrostatic pressure p at the point (x, y, z) must satisfy the differential equation

$$dp = \rho(X dx + Y dy + Z dz), \qquad [122]$$

where X, Y, and Z are the values at that point of the force applied per unit of mass to urge the liquid in directions parallel to the coördinate axes.

For, if we consider an element of the liquid $[\Delta x \, \Delta y \, \Delta z]$ (Fig. 27) whose average density is ρ_0 and whose corner next the origin has the coördinates (x, y, z), and if we denote by p_x the average pressure per unit surface on the face $PP_2P_4P_3$, by $p_x + \Delta_x p_x$ the average pressure on the face $P_1P_5P_7P_6$, and by X_0 the average applied force per unit of mass which tends to move the element in a direction parallel to the axis of x, we have, since the element is at rest,

$$p_x \Delta y \, \Delta z + \rho_0 X_0 \Delta x \, \Delta y \, \Delta z = (p_x + \Delta_x p_x) \, \Delta y \, \Delta z,$$

or

$$\rho_0 X_0 = \frac{\Delta_x p_x}{\Delta x}.$$

If the element be made smaller and smaller, the first side of the equation approaches the limit ρX, and the second side the limit $D_x p$, where p is the hydrostatic pressure, equal in all directions, at the point P.

This gives us

$$D_x p = \rho X. \qquad [123]$$

In a similar manner, we may prove that

$$D_y p = \rho Y,$$

and

$$D_z p = \rho Z;$$

whence

$$\begin{aligned} dp &= D_x p \, dx + D_y p \, dy + D_z p \, dz \\ &= \rho(X dx + Y dy + Z dz). \end{aligned}$$

If in any case of a liquid at rest the only external force applied to each particle is the attraction due to some outside mass, or to the other particles of the liquid, or to both together, X, Y, and Z are the partial derivatives with regard to x, y, and

z of a single function V, and we may write our general equation in the form

$$dp = \rho\,(D_x V \cdot dx + D_y V \cdot dy + D_z V \cdot dz) = \rho \cdot dV,$$

whence, if ρ is constant,

$$p = \rho V + \text{const.}, \qquad [124]$$

and the surfaces of equal hydrostatic pressure are also equipotential surfaces.

According to this, the free bounding surfaces of a liquid at rest under the action of gravitation only are equipotential.

EXAMPLES.

1. Prove that a particle cannot be in stable equilibrium under the attraction of any system of masses. [Earnshaw.]

2. The earth's potential function expressed in the common, kinetic, centimetre-gramme-second units is $981\, a^2/r$, for points above the surface.

3. Prove that if all the attracting mass lies within an equipotential surface S on which $V = C$, then in all space outside S the value of the potential function lies between C and 0.

4. The source of the Mississippi River is nearer the centre of the earth than the mouth is. What can be inferred from this about the slope of level surfaces on the earth?

5. If in [59] x be made equal to zero, V becomes infinite. How can you reconcile this with what is said in the first part of Section 22?

6. Are all solutions of Laplace's Equation possible values of the potential function in empty space due to distributions of matter? Assume some particular solution of this equation which will serve as the potential function due to a possible distribution and show what this distribution is.

7. If the lines of force which traverse a certain region are parallel, what may be inferred about the intensity of the force within the region?

8. The path of a material particle starting from rest at a point P and moving under the action of the attraction of a given

mass M is not in general the line of force due to M which passes through P. Discuss this statement, and consider separately cases where the lines of force are straight and where they are curved.

9. Draw a figure corresponding to Figure 17 for the case of a uniform sphere of unit radius surrounded by a concentric spherical shell of radii 2 and 3 respectively.

10. Draw with the aid of compasses traces of four of the equipotential surfaces due to two homogeneous infinite cylinders of equal density whose axes are parallel and at a distance of 5 inches apart, assuming the radius of one of the cylinders to be 1 inch and that of the other to be 2 inches.

11. Draw with the aid of compasses meridian sections of four of the equipotential surfaces due to two small homogeneous spheres of mass m and $2m$ respectively, whose centres are 4 inches apart. Can equipotential surfaces be drawn so as to lie wholly or partly within one of the spheres? What value of the potential function gives an equipotential surface shaped like the figure 8? Show that the value of the resultant force at the point where this curve crosses itself is zero.

12. A sphere of radius 3 inches and of constant density μ is surrounded by a spherical shell concentric with it of radii 4 inches and 5 inches and of density μr, where r is the distance from the centre. Compute the values of the attraction and of the potential function at all points in space and draw curves to illustrate the fact that V and D_rV are everywhere continuous and that D_r^2V is discontinuous at certain points.

13. A very long cylinder of radius 4 inches and of constant density μ is surrounded by a cylindrical shell coaxial with it and of radii 6 inches and 8 inches. The density of this shell is inversely proportional to the square of the distance from the axis, and at a point 8 inches from this axis is μ. Use the Theorem of Gauss to find the values of V, D_rV, and D_r^2V at different points on a line perpendicular to the axis of the cylinder at its middle point. If the value of the attraction at a distance of 20 inches from the axis is 10, show how to find μ.

14. Use Dirichlet's value of $D_x V$, given by equation [78], to find the attraction in the direction of the axis of x at points within a spherical shell of radii r_0 and r_1 and of constant density ρ.

15. Are there any other cases except those in which the density of the attracting matter depends only upon the distance from a plane, from an axis, or from a central point, where surfaces of equal force are also equipotential surfaces? Prove your assertion.

16. Show that the second derivative with respect to x, of the potential function due to a homogeneous sphere of density ρ and radius a, with centre at the origin, is $-\frac{4}{3}\pi\rho r$ for inside points, and $-\frac{4}{3}\pi\rho a^3(r^2 - 3x^2)/r^5$ for points without the surface. Similar expressions give the values of the second derivatives with respect to y and z. Show that the normal second derivative of V is $-\frac{4}{3}\pi\rho$ just within the surface and $+\frac{8}{3}\pi\rho$ just without. Show that the tangential second derivatives are continuous at the surface.

17. Two uniform straight wires of length l and of masses m_1 and m_2 are parallel to each other and perpendicular to the line joining their middle points, which is of length y_1. Show that the amount of work required to increase the distance between the wires to y_2 by moving one of them parallel to itself is

$$\frac{2\,m_1 m_2}{l^2}\left[y - \sqrt{l^2+y^2} - l\log\frac{\sqrt{l^2+y^2}-l}{y}\right]_{y=y_1}^{y=y_2}. \quad \text{[Minchin.]}$$

18. Show that if the earth be supposed spherical and covered with an ocean of small depth, and if the attraction of the particles of water on each other be neglected, the ellipticity of the ocean spheroid will be given by the equation,

$$e^2 = \frac{\textit{The centrifugal force at the equator}}{g}.$$

19. A spherical shell whose inner radius is r contains a mass m of gas which obeys the Law of Boyle and Mariotte. Find the law of density of the gas, the total normal pressure on the inside of the containing vessel, and the pressure at the centre.

20. If the earth were melted into a sphere of homogeneous liquid, what would be the pressure at the centre in tons per square foot? If this molten sphere instead of being homogeneous had a surface density of 2.4 and an average density of 5.6, what would be the pressure at the centre on the supposition that the density increased proportionately to the depth?

21. A solid sphere of attracting matter of mass m and of radius r is surrounded by a given mass M of gas which obeys the Law of Boyle and Mariotte. If the whole is removed from the attraction of all other matter, find the law of density of the gas and the pressure on the outside of the sphere.

22. The potential function within a closed surface S due to matter wholly outside the surface has for extreme values the extreme values upon S.

23. If the potential functions V and V' due to two systems of matter without a closed surface have the same values at all points on the surface, they will be equal throughout the space enclosed by the surface.

24. The potential function outside of a closed surface due to matter wholly within the surface has for its extreme values two of the following three quantities: zero and the extreme values upon the surface.

25. If w is harmonic in the domain T, the average value of w on any spherical surface within T is equal to the actual value at the centre of the surface. If S is a closed surface drawn in T, and if w is not constant, greater and smaller values of w are to be found on S than within it.

[Answers to some of these problems and a collection of additional problems illustrative of the text of this chapter may be found near the end of the book.]

CHAPTER III.

THE POTENTIAL FUNCTION IN THE CASE OF REPULSION.

39. Repulsion, according to the Law of Nature. Certain physical phenomena teach us that bodies may acquire, by electrification or otherwise, the property of repelling each other, and that the resulting force of repulsion between two bodies is often much greater than the force of attraction which, according to the Law of Gravitation, every body has for every other body.

Experiment shows that almost every such case of repulsion, however it may be explained physically, can be quantitatively accounted for by assuming the existence of some distribution of a kind of "matter," every particle of which is supposed to repel every other particle of the same sort according to the "Law of Nature," that is, roughly stated, with a force directly proportional to the product of the quantities of matter in the particles, and inversely proportional to the square of the distance between their centres.

In this chapter we shall assume, for the sake of argument, that such matter exists, and proceed to discuss the effects of different distributions of it. Since the law of repulsion which we have assumed is, with the exception of the opposite directions of the forces, mathematically identical with the law which governs the attraction of gravitation between particles of ponderable matter, we shall find that, by the occasional introduction of a change of sign, all the formulas which we have proved to be true for cases of attraction due to gravitation can be made useful in treating corresponding problems in repulsion.

40. Force at Any Point due to a Given Distribution of Repelling Matter. Two equal quantities of repelling matter concentrated at points at the unit distance apart are called "unit quantities" when they are such as to make the force of repulsion between them the unit force.

If the ratio of the quantity of repelling matter within a small closed surface supposed drawn about a point P, to the volume of the space enclosed by the surface, approaches the limit ρ when the surface (always enclosing P) is supposed to be made smaller and smaller, ρ is called the "density" of the repelling matter at P.

In order to find the magnitude at any point P of the force due to any given distribution of repelling matter, we may suppose the space occupied by this matter to be divided up into small elements, and compute an approximate value of this force on the assumption that each element repels a unit quantity of matter concentrated at P with a force equal to the quantity of matter in the element divided by the square of the distance between P and one of the points of the element. The limit approached by this approximate value as the size of the elements is diminished indefinitely is the value required.

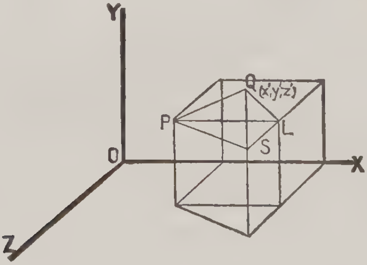

FIG. 30.

Let Q (Fig. 30), whose coördinates are x', y', z', be the corner next the origin of an element of the distribution. Let ρ be the density at Q and $\Delta x' \Delta y' \Delta z'$ the volume of the element; then the force at P due to the matter in the element is approximately equivalent to a force of magnitude $\frac{\rho\,\Delta x'\,\Delta y'\,\Delta z'}{\overline{PQ}^2}$ acting in the direction QP, or a force of magnitude $-\frac{\rho\,\Delta x'\,\Delta y'\,\Delta z'}{\overline{PQ}^2}$ acting in the direction PQ. If the coördinates of P are x, y, z, the component of this force in the direction of the positive axis of x is $\frac{-\rho\,\Delta x'\,\Delta y'\,\Delta z'(x'-x)}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{\frac{3}{2}}}$, and the force at P parallel to the axis of x due to the whole distribution of repelling matter is

$$X=-\int\int\int\frac{\rho(x'-x)\,dx'dy'dz'}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{\frac{3}{2}}}, \qquad [125_A]$$

where the triple integration is to be extended over the whole space filled with the repelling matter. For the components of the force at P parallel to the other axes we have, similarly,

$$Y=-\int\int\int\frac{\rho(y'-y)\,dx'dy'dz'}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{\frac{3}{2}}}, \qquad [125_B]$$

and

$$Z=-\int\int\int\frac{\rho(z'-z)\,dx'dy'dz'}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{\frac{3}{2}}}. \qquad [125_C]$$

If we denote by V the function

$$\int\int\int\frac{\rho\,dx'dy'dz'}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{\frac{1}{2}}}, \qquad [126]$$

which, together with its first derivatives, is everywhere finite and continuous, as we have shown in the last chapter, it is easy to see that

$$X=-D_xV, \quad Y=-D_yV, \quad Z=-D_zV, \qquad [127]$$

$$R=\sqrt{(D_xV)^2+(D_yV)^2+(D_zV)^2}, \qquad [128]$$

and that the direction-cosines of the line of action of the resultant force at P are

$$-\frac{D_xV}{R}, \quad -\frac{D_yV}{R}, \quad \text{and} \quad -\frac{D_zV}{R}. \qquad [129]$$

It follows from this (see Section 21) that the component in any direction of the force at a point P due to any distribution M of repelling matter is minus the value at P of the partial derivative of the function V taken in that direction.

The function V goes by the name of the Newtonian potential function whether we are dealing with attracting or repelling matter.

In the case of repelling matter, it is evident that the resultant force on a particle of the matter at any point tends to drive that particle in a direction which leads to points at which the potential function has a lower value, whereas in the case of gravitation a particle of ponderable matter at any point tends to move in a direction along which the potential function increases.

41. The Potential Function as a Measure of Work. It is easy to show by a method like that of Article 27 that the amount of work required to move a unit quantity of repelling matter, concentrated at a point, from P_1 to P_2, in face of the force due to any distribution M of the same kind of matter, is $V_2 - V_1$, where V_1 and V_2 are the values at P_1 and P_2 respectively of the potential function due to M. The farther P_1 is from the given distribution, the smaller is V_1, and the less does $V_2 - V_1$ differ from V_2. In fact, the value of the potential function at the point P_2, wherever it may be, measures the work which would be required to move the unit quantity of matter by any path from "infinity" to P_2.

42. Gauss's Theorem in the Case of Repelling Matter. If a quantity m of repelling matter is concentrated at a point within a closed oval surface, the resultant force due to m at any point on the surface acts toward the outside of the surface instead of towards the inside, as in the case of attracting matter.

Keeping this in mind, we may repeat the reasoning of Article 31, using repelling matter instead of attracting matter, and substituting all through the work the exterior normal for the interior normal, and in this way prove that:

If there be any distribution of repelling matter partly within and partly without a closed surface T, and if M be the whole quantity of this matter enclosed by T, and M' the quantity outside T, the surface integral over T of the component in the direction of the *exterior* normal of the force due to both M and M' is equal to $4\pi M$. If V be the potential function due to M and M', we have

$$\int D_n V \cdot ds = 4\pi M.$$

43. Poisson's Equation in the Case of Repelling Matter. If we apply the theorem of the last article to the surface of a volume element cut out of space containing repelling matter, and use the notation of Article 35, we shall find that in the case of rectangular coördinates the surface integral, taken over the element, of the component in the direction of the exterior normal is

$$\Delta x \Delta y \Delta z \left[\frac{\Delta_x X}{\Delta x} + \frac{\Delta_y Y}{\Delta y} + \frac{\Delta_z Z}{\Delta z} \right] = 4\pi\rho_0 \cdot \Delta x \Delta y \Delta z, \qquad [130]$$

where X is the average component in the positive direction of the axis of x of the force on the elementary surface $\Delta y \Delta z$, and where Y and Z have similar meanings. It is evident that if the element be made smaller and smaller, X, Y, and Z will approach as limits the components parallel to the coördinate axes of the force at P. These components are $-D_x V$, $-D_y V$, and $-D_z V$; so that if we divide [130] by $\Delta x \Delta y \Delta z$ and then decrease indefinitely the dimensions of the element, we shall arrive at the equation

$$\nabla^2 V = -4\pi\rho. \qquad [131]$$

By using successively cylinder coördinates and spherical coördinates we may prove the equations

$$\frac{1}{r} D_r(r D_r V) + \frac{1}{r^2} D_\theta^2 V + D_z^2 V = -4\pi\rho, \qquad [132]$$

and

$$\sin\theta \cdot D_r(r^2 D_r V) + \frac{D_\phi^2 V}{\sin\theta} + D_\theta(\sin\theta \cdot D_\theta V)$$
$$= -4\pi\rho r^2 \sin\theta, \qquad [133]$$

so that Poisson's Equation holds whether we are dealing with attracting or repelling matter.

44. Coexistence of Two Kinds of Active Matter. Certain physical phenomena may be most conveniently treated mathematically by assuming the coexistence of two kinds of "matter" such that any quantity of either kind repels all other matter of the same kind according to the Law of Nature, and attracts all matter of the other kind according to the same law.

Two quantities of such matter may be considered equal if, when placed in the same position in a field of force, they are subjected to resultant forces which are equal in intensity and which have the same line of action. The two quantities of matter are of the same kind if the direction of the resultant forces is the same in the two cases, but of different kinds if the directions are opposed. The unit quantity* of matter is that quantity which concentrated at a point would repel with the unit force an equal quantity of the same kind concentrated at a point at the unit distance from the first point.

It is evident from Articles 2, 14, and 40 that m units of one of these kinds of matter, if concentrated at a point (x, y, z) and exposed to the action of $m_1, m_2, m_3, \ldots m_k$ units of the same kind of matter concentrated respectively at the points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), $\ldots$ (x_k, y_k, z_k), and of $m_{k+1}, m_{k+2}, \ldots m_n$ units of the other kind of matter concentrated respectively at the points $(x_{k+1}, y_{k+1}, z_{k+1})$, $(x_{k+2}, y_{k+2}, z_{k+2})$, $\ldots$ (x_n, y_n, z_n), will be urged in the direction parallel to the positive axis of x with the force

$$X = -m\sum_{i=1}^{i=k}\frac{m_i(x_i - x)}{r_i^3} + m\sum_{i=k+1}^{i=n}\frac{m_i(x_i - x)}{r_i^3}, \qquad [134]$$

where r_i is the distance between the points (x, y, z) and (x_i, y_i, z_i).

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* With this definition of the unit of quantity, the repulsion and attraction force unit is identical with the absolute kinetic force unit.

If we agree to distinguish the two kinds of matter from each other by calling one kind "positive" and the other kind "negative," it is easy to see that if every m which belongs to positive matter be given the plus sign and every m which belongs to negative matter the minus sign, we may write the last equation in the form

$$X = -m \sum_{i=1}^{i=n} \frac{m_i (x_i - x)}{r_i^3}. \qquad [135]$$

The result obtained by making m in [135] equal to unity is called the force at the point (x, y, z).

In general, m units of either kind of matter concentrated at the point (x, y, z), and exposed to the action of any continuous distribution of matter, will be urged in the positive direction of the axis of x by the force

$$X = -m \int\int\int \frac{\rho (x' - x)\, dx' dy' dz'}{[(x' - x)^2 + (y' - y)^2 + (z' - z)^2]^{\frac{3}{2}}}; \qquad [136]$$

in this expression, ρ, the density at (x', y', z'), is to be taken positive or negative according as the matter at the point is positive or negative: m is to have the sign belonging to the matter at the point (x, y, z): and the limits of integration are to be chosen so as to include all the matter which acts on m.

With the same understanding about the signs of m and of ρ, it is clear that the force which urges in any direction s, m units of matter concentrated at the point (x, y, z) is equal to $-m \cdot D_s V$, where V is the everywhere finite, continuous, and single-valued function

$$\int\int\int \frac{\rho\, dx'\, dy'\, dz'}{[(x' - x)^2 + (y' - y)^2 + (z' - z)^2]^{\frac{1}{2}}};$$

and that mV measures the amount of work required to bring up from "infinity" by any path to its present position the m units of matter now at the point (x, y, z).

If we call the resultant force which would act on a unit of positive matter concentrated at the point P "the force at P," it is clear that if any closed surface T be drawn in a field of force due to any distribution of positive and negative matter so as to include a quantity of this matter algebraically equal to Q, the surface integral taken over T of the component in the direction of the exterior normal of the force at the different points of the surface is equal to $4\pi Q$.

It will be found, indeed, that all the equations and theorems given earlier in this chapter for the case of one kind of repelling matter may be used unchanged for the case where positive and negative matter coexist, if we only give to ρ and m their proper signs.

It is to be noticed that Poisson's Equation is applicable whether we are dealing with attracting matter or repelling matter, or positive and negative matter existing together.

EXAMPLES.

1. Show that the extreme values of the potential function outside a closed surface S, due to a quantity of matter algebraically equal to zero within the surface, are its extreme values on S.

2. Show that if the potential function due to a quantity of matter algebraically equal to zero and shut in by a closed surface S has a constant value all over the surface, then this constant value must be zero.

3. Show that if the function w, which is harmonic everywhere outside the finite closed surface S, vanishes at infinity, and if r represents the distance from any fixed point, $\underset{r=\infty}{\text{limit}} (r^2 \cdot D_r w)$ is finite.

CHAPTER IV.

SURFACE DISTRIBUTIONS.—GREEN'S THEOREM.

45. Force due to a Closed Shell of Repelling Matter. If a quantity of very finely-divided repelling matter be enclosed in a box of any shape made of indifferent material, it is evident from [127] and from the principles of Section 38 that if the volume of the box is greater than the space occupied by the repelling matter, the latter will arrange itself so that its free surface will be equipotential with regard to all the active matter in existence, taking into account any there may be outside the box as well as that inside. It is easy to see, moreover, that we shall have a shell of matter lining the box and enclosing an empty space in the middle.

That any such distribution as that indicated in the subjoined diagram is impossible follows immediately from the reasoning of Section 37. For *ABC* and *DEF* are parts of the same equipotential free surface of the matter. If we complete this surface by the parts indicated by the dotted lines, we shall enclose a space void of matter and having therefore throughout a value of the potential function equal to that on the bounding

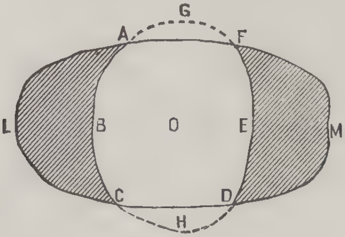

Fig. 31.

surface. But in this case all points which can be reached from O by paths which do not cut the repelling matter must be at the same potential as O, and this evidently includes all space not actually occupied by the repelling matter; which is absurd.

Let us consider, then (see Fig. 32), a closed shell of repelling matter whose inner surface is equipotential, so that at every point of the cavity which the shell shuts in, the resultant force, due to the matter of which the shell is composed and to any outside matter there may be, is zero.

Let us take a small portion ω of the bounding surface of the cavity as the base of a tube of force which shall intercept an area ω' on an equipotential surface which cuts it just outside the outer surface of the shell, and let us apply Gauss's Theorem to the box enclosed by ω, ω', and the tube of force. If F' is the average value of the resultant force on ω', the only part of the surface of the box which yields anything to the surface integral of normal force, we have

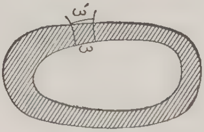

FIG. 32.

$$F'\omega' = 4\pi m,$$

where m is the quantity of matter within the box. If we multiply and divide by ω, this equation may be written

$$F' = \frac{4\pi m}{\omega} \cdot \frac{\omega}{\omega'}. \qquad [137]$$

If ω be made smaller and smaller, so as always to include a given point A, ω' as it approaches zero will always include a point B on the line of force drawn through A, and F' will approach the value F of the resultant force at B.

The shell may be regarded as a thick layer spread upon the inner surface, and in this case the limit of $\frac{m}{\omega}$ may be considered the value at A of the rate at which the matter is spread upon the surface. If we denote this limit by σ, we shall have

$$F = 4\pi\sigma \cdot \underset{\omega \doteq 0}{\text{limit}} \left(\frac{\omega}{\omega'}\right). \qquad [138]$$

If B be taken just outside the shell, and if the latter be very thin, $\underset{\omega \doteq 0}{\text{limit}} \left(\frac{\omega}{\omega'}\right)$ evidently differs little from unity; and we see that the resultant force at a point just outside the outer surface of a shell of matter, whose inner surface is equipotential, becomes more and more nearly equal to 4π times the quantity of matter per unit of surface in the distribution at that point as the shell becomes thinner and thinner.

The reader may find out for himself, if he pleases, whether or not the line of action of the resultant force at a point just outside such a shell as we have been considering is normal to the shell.

It is to be carefully noticed that the inner surface of a closed shell need not be equipotential unless the matter composing the shell is finely divided and free to arrange itself at will.

When the shell is thin, and we regard it as formed of matter spread upon its inner surface, σ is called the "surface density" of the distribution, and its value at any point of the inner surface of the shell may be regarded as a measure of the amount of matter which must be spread upon a unit of surface if it is to be uniformly covered with a layer of thickness equal to that of the shell at the point in question.

46. Surface Distributions. It often becomes necessary in the mathematical treatment of physical problems, on the assumption of the existence of a kind of repelling matter or agent, to imagine a finite quantity of this agent *condensed on a surface* in a layer so thin that for practical purposes we may leave the thickness out of account. If a shell like that considered in the last section could be made thinner and thinner by compression while the quantity of matter in it remained unchanged, the volume density (ρ) of the shell would grow larger and larger without limit, and σ would remain finite. A distribution like this, which is considered to have *no* thickness, is called a surface distribution.

The value at a point P of the potential function due to a superficial distribution of surface density σ is the surface integral, taken over the distribution, of $\frac{\sigma}{r}$, where r is the distance from P.

It is evident that as long as P does not lie exactly in the distribution, the potential function and its derivatives are always finite and continuous, and the force at any point in any direction may be found by differentiating the potential function partially with regard to that direction.

If ρ were infinite, the reasoning of Article 22 would no longer apply to points actually in the active matter, and it is worth our while to prove that in the case of a surface distribution where σ is everywhere finite, the value at a point P of the potential function due to the distribution remains finite, as P is made to move normally through the surface at a point of finite curvature.

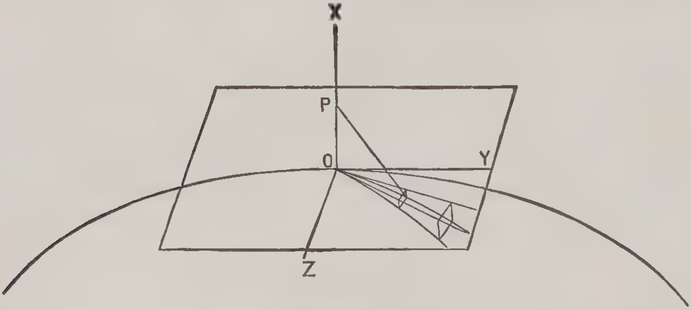

FIG. 33.

To show this, take the point O (Fig. 33), where P is to cut the surface, as origin, and the normal to the surface at O as the axis of x, so that the other coördinate axes shall lie in the tangent plane.

If the curvature in the neighborhood of O is finite, it will be possible to draw on the surface about O a closed line such that for every point of the surface within this line the normal will make an acute angle with the axis of x.

For convenience we will draw the closed line of such a shape that its projection on the tangent plane shall be a circle whose centre is at O and whose radius is U, and we will cut the area shut in by this line into elements of such shape that their projections upon the tangent plane shall divide the circle just mentioned into elements bounded by concentric circumferences drawn at radial intervals of Δu, and by radii drawn at angular distances of $\Delta\phi$.

If x, 0, 0 are the coördinates of the point P, x', y', z' those of a point of one of the elements of the area shut in by the closed line, and α the angle which the normal to the surface at this point makes with the axis of x, the size of the surface element is approximately $\frac{u\,\Delta u\,\Delta\phi}{\cos\alpha}$, where $u^2 = z'^2 + y'^2$, and the value at P of the potential function due to that part of the surface distribution shut in by the closed line is

$$V_1 = \int_0^{2\pi} d\phi \int_0^U \frac{\sigma u\,du}{\cos\alpha\sqrt{(x-x')^2+u^2}}. \qquad [139]$$

The quantity

$$\frac{\sigma u}{\cos\alpha\sqrt{(x-x')^2+u^2}} = \frac{\sigma\sec\alpha}{\sqrt{1+\left(\frac{x-x'}{u}\right)^2}}$$

is always finite, for, whatever the value of the quantity under the radical sign in the last expression may be when x, x', and u are all zero, it cannot be less than unity, and therefore V_1 must be finite even when P moves down the axis of x to the surface itself.

If V and V_2 are the values at P of the potential functions due respectively to all the existing acting matter and to that part of this matter not lying on the portion of the surface shut in by our closed line, we have $V = V_1 + V_2$, and, since P is a point outside the matter which gives rise to V_2, the latter is finite; so that V is finite.

The reader who wishes to study the properties of the derivatives of the potential function, and their relations to the force components at points actually in a surface distribution, will find the whole subject treated in the first part of Riemann's *Schwere, Electricität und Magnetismus.*

Using the notation of this section, it is easy to write down definite integrals which represent the values of the potential function at two points on the same normal, one on one side of a superficial distribution, and at a distance a from it, and the other on the other side at a like distance, and to show that the difference between these integrals may be made as small as we like by choosing a small enough. This shows that the value of the potential function at a point P changes continuously, as P moves normally through a surface distribution of finite superficial density. If matter could be concentrated upon a geometric line, so that there should be a finite quantity of matter on the unit of length of the line, or if a finite quantity of matter could be really concentrated at a point, the resulting potential function would be infinite on the line itself, and at the point.

47. The Normal Force at Any Point of a Surface Distribution. In the case of a strictly superficial distribution on a closed surface where the repelling matter is free to arrange itself at will, the inner surface of the matter (and hence the outer surface, which is coincident with it) is equipotential, and the resultant force at a point B just outside the distribution is normal to the surface and numerically equal to 4π times the surface density at B. This shows that the derivative of the potential function in the direction of the normal to the surface has values on opposite sides of the surface differing by $4\pi\sigma$, and at the surface itself cannot be said to have any definite value.

It is easy, however, to find the force with which the repelling matter composing a superficial distribution is urged outwards. For, take a small element ω of the surface as the base of a tube of force, and apply Gauss's Theorem to a box shut in by the surface of distribution, the tube of force, and a portion ω' of an equipotential surface drawn just outside the distribution. Let F and F' be the average forces at the points of ω and ω' respectively, then the surface integral of normal forces taken over the box is $F'\omega' - F\omega$, and this, since the only active matter is concentrated on the surface of the box (see Section 31), is equal to $2\pi\sigma_0\omega$, where σ_0 is the average surface density at the points of the element ω. This gives us

$$F = F'\frac{\omega'}{\omega} - 2\pi\sigma_0.$$

Now let the equipotential surface of which ω' is a part be drawn nearer and nearer the distribution; then

$$\lim\frac{\omega'}{\omega} = 1, \quad \lim F' = 4\pi\sigma_0, \quad \text{and} \quad F = 2\pi\sigma_0.$$

F is the average force which would tend to move a unit quantity of repelling matter concentrated successively at the different points of ω in the direction of the exterior normal, but the actual distribution on ω is $\omega\sigma_0$, so that this matter presses on the medium which prevents it from escaping with the force $2\pi\sigma_0^2\omega$; and, in general, the pressure exerted on the resisting medium which surrounds a surface distribution of repelling matter is at any point $2\pi\sigma^2$ per unit of surface, where σ is the surface density of the distribution at the point in question.

We may imagine a superficial distribution of matter which is fixed, instead of being free to arrange itself at will. In this case the surface of the matter will not be in general equipotential, but, if we apply Gauss's Theorem to a box shut in by a slender tube of force traversing the distribution, and by two surfaces drawn parallel to the distribution and close to it, one on one side and one on the other, we may prove that the normal component of the force at a point just outside the distribution differs by $4\pi\sigma$ from the normal component, in the same sense, of the force at a point just inside the distribution on the line of force which passes through the first point.

It is sometimes convenient to denote the "charge" on a small area about a point P on a surface distribution by A', and the rest of the distribution by A'', and to consider separately the effects of A' and A''. If P_1 and P_2 are points on the normal to the surface drawn through P and near the surface on opposite sides of it; if N_1', N_1'' are the components in the direction PP_1 of the forces at P_1 due to A' and A'' respectively, and if N_2', N_2'' are the corresponding components at P_2 in the direction PP_2, then if P_1 and P_2 approach P,

$$\lim\left[N_1' + N_1'' + N_2' + N_2''\right] = 4\pi\sigma,$$

where σ is the density of the distribution at P. The force due to A'' changes continuously as P_1 moves toward P_2, however small A' may be, so that

$$\lim N_1'' = -\lim N_2'' \text{ and } \lim (N_1' + N_2') = 4\pi\sigma,$$

and, by choosing A' small enough, we may make N_1' to differ in numerical value as little as we please from $\lim N_2'$ or from $2\pi\sigma$.

If the surface distribution is equipotential, and if it shuts in a region of no force, then if P_1 is in this region, $N_1' = -N_1''$, so that N_1'' and N_2'' can be made to differ as little as one pleases in numerical value from $2\pi\sigma$ by making A' small enough. Let the element of area covered by A' be ω and the surface density of the charge on it σ, then the force with which A' is urged in a direction normal to the surface by A'' is $\omega\sigma \cdot 2\pi\sigma$ within an infinitesimal of higher order than ω. That is, whatever the sign of σ, the surface distribution may be said to urge the surrounding medium outwards with a pressure in force units per unit of area which at P has the value $2\pi\sigma^2$, as we have already seen.

It is easy to show that even if the surface distribution is not equipotential the components at P_1 and P_2 of the force in any fixed direction parallel to the surface approach the same limit as P_1 and P_2 approach P.

At any point P of an equipotential surface covered with a superficial distribution of density σ the normal second derivative of V has a discontinuity* of $4\pi\sigma\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$ where R_1 and R_2 are the radii of curvature at P of two mutually perpendicular normal sections of the equipotential surface.

48. Green's Theorem. Before proving a very general theorem due to Green,† of which what we have called Gauss's Theorem is a special case, we will show that if S is any closed surface and U a function of x, y, and z, which for every point inside S is continuous, and single-valued,

$$\iiint D_x U \cdot dx\,dy\,dz = \int U \cdot D_n x \cdot ds, \qquad [140]$$

where the first integral is to include all the space shut in by S, and the second is to be taken over the whole surface, and where $D_n x$ represents the derivative of x taken in the direction of the *exterior* normal.

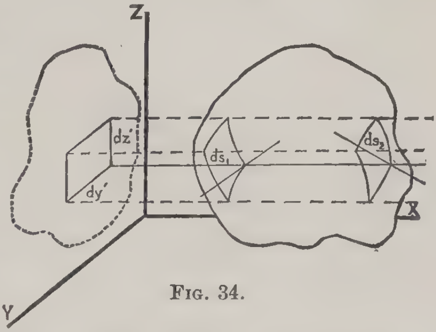

FIG. 34.

To prove this, choose the coördinate axes so that S shall lie in the first octant, and divide the space inside the contour of the projection of S on the plane yz into elements of size $dy\,dz$. On each of these elements erect a right prism cutting S twice or some other even number of times. Let us call the values of U at the successive points where the edge nearest the axis of x of any one of these prisms cuts S; $U_1, U_2, U_3, \cdots U_{2n}$ respectively; the angles which this edge makes with exterior normals drawn to S at these points, $\alpha_1, \alpha_2, \alpha_3, \cdots \alpha_{2n}$; and the elements which the prism cuts from the surface S; $ds_1, ds_2, ds_3, \cdots ds_{2n}$. It is evident that wherever a line perpendicular to the plane yz cuts *into* S, the corresponding value of α is obtuse and its cosine negative, but wherever such a line cuts *out* of S, the corresponding value of α is acute and its cosine positive.

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* C. Neumann, *Math. Ann.* 1880. Th. Horn, *Zeitschr. f. Math. u. Phys.* 1881.

† George Green, *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism.* Nottingham, 1828.

Keeping this in mind, we shall see that although the base of a prism is the common projection of all the elements which it cuts from S, and in absolute value is approximately equal to any one of these multiplied by the corresponding value of $\cos \alpha$, yet, since $dx\,dy$, ds_1, ds_2, etc., are all positive areas and some of the cosines are negative, we must write, if we take account of signs,

$$dy\,dz = -ds_1 \cos\alpha_1 = +ds_2 \cos\alpha_2 = -ds_3 \cos\alpha_3 = \cdots.$$

If the indicated integration with regard to x in the left-hand member of [140] be performed and the proper limits introduced, we shall have

$$\int\int\int D_x U\,dx\,dy\,dz = \int\int dy\,dz[-U_1 + U_2 - U_3 + U_4 - \cdots], \quad [141]$$

where the double sign of integration directs us to form a quantity corresponding to that in brackets for every prism which cuts S, to multiply this by the area of the base of the prism, and to find the limit of the sum of all the results as the bases of the prisms are made smaller and smaller.

Since we may substitute for $dy\,dz$ any one of its approximate values given above, we may write the quantity within the brackets

$$U_1 \cos\alpha_1\,ds_1 + U_2 \cos\alpha_2\,ds_2 + U_3 \cos\alpha_3\,ds_3 + \cdots,$$

and this shows that the double integral is equivalent to the surface integral, taken over the whole of S, of $U\cos\alpha$, whence we may write

$$\int\int\int D_x U \cdot dx\,dy\,dz = \int U \cos\alpha\,ds, \quad [142]$$

where the first integral is to be taken all through the space shut in by S, and the second over the whole surface.

Let P or (x, y, z) be any point of S, α, β, and γ the angles which the exterior normal drawn at P to S makes with the coördinate axes, and P' a point on this normal at a distance Δn from P. The coördinates of P' are

$$x + \Delta n \cdot \cos\alpha, \quad y + \Delta n \cdot \cos\beta, \quad z + \Delta n \cdot \cos\gamma,$$

and if $W = f(x, y, z)$ be any continuous function of the space coördinates,

$$\begin{aligned} W_P &= f(x, y, z), \\ W_{P'} &= f(x + \Delta n \cos\alpha,\ y + \Delta n \cos\beta,\ z + \Delta n \cos\gamma) \\ &= f(x, y, z) + \Delta n \cos\alpha \cdot D_x f + \Delta n \cos\beta \cdot D_y f \\ &\qquad + \Delta n \cos\gamma\, D_z f + (\Delta n)^2 Q, \end{aligned}$$

and

$$\frac{W_{P'} - W_P}{PP'} = \cos\alpha \cdot D_x f + \cos\beta \cdot D_y f + \cos\gamma \cdot D_z f + \Delta n \cdot Q,$$

whence

$$\lim \frac{W_{P'} - W_P}{PP'} = D_n W_P = \cos\alpha\, D_x f + \cos\beta\, D_y f + \cos\gamma\, D_z f. \quad [143]$$

If, as a special case, $W = x$, we have $D_n x = \cos\alpha$; so that [142] may be written

$$\iiint D_x U \cdot dx\, dy\, dz = \int U D_n x \cdot ds, \qquad [144]$$

which we were to prove.*

Green's Theorem, which follows very easily from this result, may be stated in the following form:

If U and V are any two functions of the space coördinates which together with their first derivatives with respect to these coördinates are finite, continuous, and single-valued throughout the space shut in by any closed surface S, then, if n refers to an exterior normal,

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* This theorem has been virtually proved already in Sections 29 and 33.

$$\iiint (D_xU\cdot D_xV + D_yU\cdot D_yV + D_zU\cdot D_zV)\,dx\,dy\,dz$$

$$= \int U\cdot D_nV\cdot ds - \iiint U\cdot\nabla^2V\cdot dx\,dy\,dz \qquad [145]$$

$$= \int V\cdot D_nU\cdot ds - \iiint V\cdot\nabla^2U\cdot dx\,dy\,dz, \qquad [146]$$

where the triple integrals include all the space within S and the single integrals include the whole surface.

Since $\quad D_xU\cdot D_xV = D_x(U\cdot D_xV) - U\cdot D_x^2V,$

we have $\quad \iiint D_xU\cdot D_xV\cdot dx\,dy\,dz$

$$= \iiint D_x(U\cdot D_xV)\,dx\,dy\,dz - \iiint U\cdot D_x^2V\cdot dx\,dy\,dz\,;$$

but, from [144],

$$\iiint D_x(U\cdot D_xV)\,dx\,dy\,dz = \int U\cdot D_xV\cdot D_nx\cdot ds,$$

whence $\quad \iiint (D_xU\cdot D_xV)\,dx\,dy\,dz$

$$= \int U\cdot D_xV\cdot D_nx\cdot ds - \iiint U\cdot D_x^2V\cdot dx\,dy\,dz. \quad [147]$$

If we form the two corresponding equations for the derivatives with regard to y and z, and add the three together, we shall obtain an expression which, by the use of [143], reduces immediately to [145]. Considerations of symmetry give [146].

If we subtract [146] from [145], we get

$$\iiint (U\cdot\nabla^2V - V\cdot\nabla^2U)\,dx\,dy\,dz$$

$$= \int (U\cdot D_nV - V\cdot D_nU)\,ds. \qquad [148]$$

In applying Green's Theorem to such spaces as those marked T_0 in the adjoining diagrams, it is to be noticed that the walls of the cavities, marked S' and S'', as well as the surfaces, marked S, form parts of the boundaries of the spaces, and that the surface integrals, which the theorem declares must be taken over the complete boundaries of the spaces, are to be extended over S' and S'' as well as over S. We must remember, however, that an exterior normal to T_0 at S' points *into* the cavity C'.

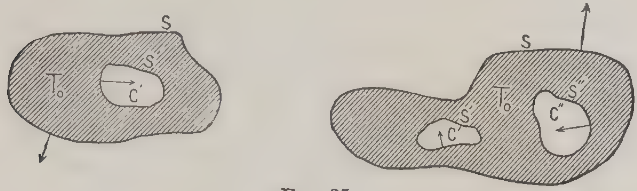

FIG. 35.

If U and V both satisfy Laplace's Equation, the second member of [148] is equal to zero.

If within the closed surface S the functions λ, U, and V are continuous, and if the first derivatives of U and V are continuous (the first derivatives of λ and the second derivatives of U and V being finite),

$$\int\int\int \lambda (D_xU \cdot D_xV + D_yU \cdot D_yV + D_zU \cdot D_zV)\, dx\, dy\, dz$$

$$= \int\int \lambda U \cdot D_nV\, dS - \int\int\int U [D_x(\lambda \cdot D_xV)$$

$$+ D_y(\lambda \cdot D_yV) + D_z(\lambda \cdot D_zV)]\, dx\, dy\, dz \qquad [149]$$

$$= \int\int \lambda V \cdot D_nU\, dS - \int\int\int V [D_x(\lambda \cdot D_xU)$$

$$+ D_y(\lambda \cdot D_yU) + D_z(\lambda \cdot D_zU)]\, dx\, dy\, dz.$$

Special Cases under Green's Theorem. Applications.

I. If in [145] we put $U = 1$, we learn that if V is any function which within and on the closed surface S is finite and continuous, together with its derivatives of the first order, the surface integral of $D_n V$ taken over S is equal to the volume integral of $\nabla^2 V$ taken through the space shut in by S. If V happens to satisfy Laplace's Equation within S, the surface integral is equal to zero. This result should be compared with Gauss's Theorem, treated in Section 31.

II. If in [145] we make U equal to V, the potential function due to any distribution of matter, and assume that, in the general case, some of this matter is spread superficially on a surface S (or on a number of such surfaces), we may shut in S by two other surfaces, S_1 and S_2, parallel and very close to it. We may then apply Green's Theorem to so much of the space within a spherical surface, with centre at some convenient fixed point and radius r large enough to include the whole distribution, as does not lie between S_1 and S_2. This gives

$$\begin{aligned}\iiint [(D_x V)^2 + (D_y V)^2 + (D_z V)^2]\, dx\, dy\, dz \\ = \int V \cdot D_n V\, dS' - \int V \cdot D_{n_1} V\, dS_1 - \int V \cdot D_{n_2} V\, dS_2 \\ - \iiint V \cdot \nabla^2 V\, dx\, dy\, dz,\end{aligned}$$

where the first surface integral is to be extended over the spherical surface, the second over S_1, and the third over S_2, it being understood that n_1 represents a normal to S_1 taken in the direction away from S, and n_2 a normal to S_2 taken in the direction away from S. Since V is continuous at S, while its normal derivatives are discontinuous in the manner indicated by the equation $D_{n_1} V + D_{n_2} V = -4\pi\sigma$, the limit of the sum of the two surface integrals taken over S_1 and S_2 as these surfaces approach S is $4\pi \int V\sigma\, ds$. The value of the first surface integral is equal to $4\pi r^2$ times the average value of $V \cdot D_r V$ on the surface; and, if this be written in the form 4π

[average value of $V(r^2D_rV)$], it is evident that the integral approaches zero as the radius r is made infinite, so that the field of the triple integrals may embrace all space. Since $\nabla^2V=-4\pi\rho$, the whole second member of the equation represents $4\pi\lim\sum V\Delta m$ extended over all the distribution, and this is 8π times the intrinsic energy of the distribution. The first member of the equation represents the volume integral of the square of the resultant force extended over all space. We may write this result in the form

$$W=\frac{1}{8\pi}\int\int_{\infty}\int R^2\,dx\,dy\,dz. \qquad [150]$$

III. If in [145] we make $U=V=u$, any function which within the closed surface S satisfies the equation $\nabla^2u=0$, we shall have

$$\int\int\int[(D_xu)^2+(D_yu)^2+(D_zu)^2]\,dxdydz=\int u\cdot D_nu\cdot dS. \qquad [151]$$

IV. If in [148] V is the potential function due to two distributions of active matter, M_1 inside the closed surface S and M_2 outside it, and if $U=\frac{1}{r}$, where r is the distance of the point (x, y, z) from a fixed point O, we must consider separately the two cases where O is respectively without S and within S.

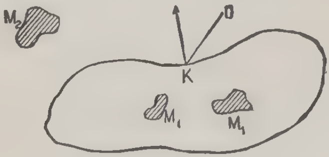

FIG. 36.

A. If O is without S, $\nabla^2\left(\frac{1}{r}\right)=0$ for points within the surface. Also, $\nabla^2V=-4\pi\rho$, so that

$$\int \frac{D_n V}{r} dS - \int V \cdot D_n\left(\frac{1}{r}\right) dS = -4\pi \int\int\int \frac{\rho}{r}\, dx\, dy\, dz.$$

The triple integral is evidently equal to the value at the point O of the potential function due to M_1 alone. If we call this V_1, and notice (see [143]) that $D_n r$ at any point of S is the cosine of the angle δ between r and the exterior normal to S, we have

$$\int \frac{D_n V}{r} dS + \int \frac{V \cos\delta}{r^2} dS = -4\pi V_1. \qquad [152]$$

If S is a surface equipotential with respect to the joint action of M_1 and M_2, and if we denote by V_s the constant value of V on S, we have

$$\int \frac{D_n V}{r} dS + V_s \int \frac{\cos\delta}{r^2} dS = -4\pi V_1,$$

and it is easy to show, by the reasoning used in Section 31, that $\int \frac{\cos\delta}{r^2} dS = 0$, whence

$$V_1 = -\frac{1}{4\pi}\int \frac{D_n V}{r} dS. \qquad [153]$$

B. If O is a point inside S, whether or not it is within M_1, and if S is equipotential with respect to the action of M_1 and M_2, we will surround O by a small spherical surface S' of radius r' and apply [148] to the space lying inside S and without the spherical surface. In doing so, it is to be noticed that S' forms part of the boundary of the region we are dealing with, and that an exterior normal to the region at S' will be an interior normal of the sphere.

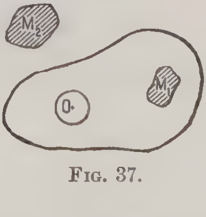

FIG. 37.

Since for all points of the region we are considering $\nabla^2\left(\frac{1}{r}\right)=0$, we have

$$\int \frac{D_n V}{r}\, dS - \int \frac{D_{r'} V}{r'}\, dS' - V_s \int D_n\left(\frac{1}{r}\right) dS + \int V \cdot D_{r'}\left(\frac{1}{r'}\right) dS'$$
$$= -4\pi \int\int\int \frac{\rho}{r}\, dx\, dy\, dz;$$

or, since $dS' = r'^2 d\omega'$, where $d\omega'$ is the area which the elementary cone the base of which is dS' and the vertex O intercepts on the sphere of unit radius drawn about O,

$$\int \frac{D_n V}{r}\, dS + V_s \int \frac{\cos\delta}{r^2}\, dS - r' \int D_{r'} V \cdot d\omega' - \int V d\omega'$$
$$= -4\pi \int\int\int \frac{\rho}{r}\, dx\, dy\, dz. \qquad [154]$$

It is easily proved, by the reasoning of Section 31, that

$$\int \frac{\cos\delta}{r^2}\, dS = 4\pi,$$

and it is clear that if r' be made smaller and smaller, the third integral of [154] approaches the limit zero. If V' is the average value of V on the surface S',

$$\int V d\omega' = V' \int d\omega' = V' 4\pi;$$

and as r' is made smaller and smaller this approaches the value $4\pi V_O$, where V_O is the value of V at O. The value, when r' is zero, of the triple integral is evidently V_1, and we have

$$\int \frac{D_n V}{r}\, dS + 4\pi V_s - 4\pi V_O = -4\pi V_1.$$

If V_2 is the value at O of the potential function due to M_2 alone, $V_O = V_1 + V_2$, so that

$$V_s - V_2 = -\frac{1}{4\pi}\int \frac{D_n V}{r}\, dS. \qquad [154_A]$$

If S is not equipotential with respect to the action of M_1 and M_2, we have

$$4\pi V_2 = \int \frac{D_n V}{r}\, dS - \int V \cdot D_n\left(\frac{1}{r}\right) dS. \qquad [154_B]$$

V. If in [148] we make $U = \frac{1}{r}$, where r is the distance of the point (x, y, z) from a fixed point O, and if $V = v$, any function harmonic everywhere within the closed surface S, we shall have

$$-4\pi v = \int v D_n\left(\frac{1}{r}\right) dS - \int \frac{D_n v}{r}\, dS, \qquad [155_A]$$

if O is within S, and

$$\int \frac{D_n v}{r}\, dS = \int v \cdot D_n\left(\frac{1}{r}\right) dS, \qquad [155_B]$$

if O is outside S.

VI. The closed surface S encloses a region T_1 and excludes the rest of space, T_2. A function V is continuous and has finite first and second derivatives everywhere in the field of Green's Theorem. The first derivatives are everywhere continuous except at certain surfaces, S_1' in T_1 and S_2' in T_2, where the tangential derivatives are continuous, and the normal derivatives discontinuous in the manner indicated by the equation

$$D_{n_1} V + D_{n_2} V = -\delta.$$

At infinity V vanishes like the Newtonian Potential Function due to a finite distribution of matter. If U is the reciprocal of the distance from a fixed point O, and if we apply Green's Theorem to U and V, using successively as fields, T_1 when O is in T_2, T_1 when O is in T_1, T_2 when O is in T_1, and

T_2 when O is in T_2, and representing by n a normal to S pointing *into* T_2 in all cases, we learn that the expression

$$-\left\{\iint \frac{V\cos(n,r)}{r^2}\,dS + \iint \frac{D_nV}{r}\,dS\right\}$$

or

$$\left\{\iint VD_n\left(\frac{1}{r}\right)dS - \iint \frac{D_nV}{r}\,dS\right\}$$

is equal respectively to

$$\iint \frac{\delta}{r}\,dS_1' - \iiint \frac{\nabla^2 V}{r}\,d\tau_1 \qquad [156_A]$$

$$-4\pi V_{O_1} + \iint \frac{\delta}{r}\,dS_1' - \iiint \frac{\nabla^2 V}{r}\,d\tau_1 \qquad [156_B]$$

$$-\iint \frac{\delta}{r}\,dS_2' + \iiint \frac{\nabla^2 V}{r}\,d\tau_2, \qquad [156_C]$$

$$4\pi V_{O_2} - \iint \frac{\delta}{r}\,dS_2' + \iiint \frac{\nabla^2 V}{r}\,d\tau_2. \qquad [156_D]$$

If there is no surface S' at which the normal derivative of V is discontinuous, and if V satisfies Laplace's Equation everywhere within S, the expression

$$\frac{1}{4\pi}\iint\left(\frac{V\cos(n,r)}{r^2} + \frac{D_nV}{r}\right)dS$$

is equal to zero or to the value of V at O according as O is without or within S.

If, now, S is a spherical surface of radius a, and if O_1 is distant l_1 from the centre C, the distance from C of O_2, the inverse point of O_1 with respect to S may be denoted by l_2, where $l_1 l_2 = a^2$. If r_1 and r_2 represent the distances of any point P from O_1 and O_2 respectively, then, if P lies on S,

$$r_1/r_2 = l_1/a,$$

$$l_1^2 = r_1^2 + a^2 - 2\,r_1 a \cdot \cos(r_1, n),$$

$$l_2^2 = r_2^2 + a^2 - 2\,r_2 a \cdot \cos(r_2, n),$$

and
$$\frac{\cos(r_1, n)}{r_1^2} - \frac{a}{l_1} \cdot \frac{\cos(r_2, n)}{r_2^2} = \frac{a^2 - l_1^2}{a r_1^3}.$$

In this case,

$$4\,\pi V_{O_1} = \iint \frac{D_n V}{r_1} dS + \iint \frac{V \cos(n, r_1)}{r_1^2} dS$$

and
$$0 = \iint \frac{D_n V}{r_2} dS + \iint \frac{V \cos(n, r_2)}{r_2^2} dS$$

$$= \frac{l_1}{a} \iint \frac{D_n V}{r_1} dS + \iint \frac{V \cos(n, r_2)}{r_2^2} dS,$$

so that it is easy to eliminate $D_n V$ by multiplying the second equation by a / l_1 and subtracting the members from those of the first equation. The result is

$$4\,\pi V_{O_1} = \iint V \left(\frac{\cos(r_1, n)}{r_1^2} - \frac{a}{l_1} \cdot \frac{\cos(r_2, n)}{r_2^2} \right) dS,$$

or
$$V_{O_1} = \frac{1}{4\,\pi a} \iint \frac{V(a^2 - l_1^2)}{r_1^3} dS \qquad [157]$$

$$= \frac{1}{4\,\pi a} \iint \frac{V(a^2 - l_1^2)\, dS}{[a^2 + l_1^2 - 2\, a l_1 \cos(a, l_1)]^{3/2}}.$$

This integral determines V at every point within S when its value is given at every point on S. If O_1 is at the centre of S, $l_1 = 0$, and $r_1 = a$, so that $V_{O_1} = \frac{1}{4\,\pi a^2} \iint V\, dS$, or the average value on a spherical surface S, of a function V, harmonic within and on S is the value of V at the centre of S. It follows from this that a function which is harmonic about a point O cannot have at O either a maximum or a minimum value.

If a function V is constant on any analytic surface S, is harmonic without S, and if it vanishes at infinity like a Newtonian Potential Function,

$$V_{O_2} = -\frac{1}{4\pi}\int\int\frac{D_n V}{r}\,dS$$

and V is the potential function in outer space due to a superficial distribution on S of surface density $-D_nV/4\pi$.

VII. A function V has the value zero everywhere on the closed surface S_1, and the constant value C on the closed surface S_2, shut in by S_1. In the space T, between S_1 and S_2, V is harmonic. If we apply Green's Theorem, in T, to V and to the reciprocal of the distance from any point O in T, we learn that

$$V_{\text{at } O} = \frac{1}{4\pi}\int\int\frac{D_n V}{r}\,dS_1 + \frac{1}{4\pi}\int\int\frac{D_n V}{r}\,dS_2,$$

where both normals point out of T.

V is, therefore, the potential function due to surface distributions on S_1 and S_2 numerically equal to $D_nV/4\pi$ at every point.

VIII. If the closed surface S shuts in a region T, and if the functions V and V', which are equal at every point of S, are finite and continuous with their derivatives of the first order at every point of T, and if within T, V does and V' does not satisfy Laplace's Equation, then the integral

$$Q_V \equiv \int\int\int[(D_xV)^2 + (D_yV)^2 + (D_zV)^2]\,dx\,dy\,dz,$$

extended throughout T is less than the corresponding integral

$$Q_{V'} \equiv \int\int\int[(D_xV')^2 + (D_yV')^2 + (D_zV')^2]\,dx\,dy\,dz.$$

If we write $V' = V + u$, u vanishes at every point of S, but is in general different from zero.

$$\begin{aligned}
&Q_{V'}\\
&=Q_V+Q_u+2\int\int\int[D_xu\cdot D_xV+D_yu\cdot D_yV+D_zu\cdot D_zV]dxdydz\\
&=Q_V+Q_u+2\int\int u\cdot D_nVdS-2\int\int\int u\cdot\bar{\nabla}^2Vdxdydz\\
&=Q_V+Q_u.
\end{aligned}$$

Now, since the integrands of Q_u and Q_V are made up of squares, and since neither u nor V are constants, both Q_u and Q_V are positive, so that $Q_{V'}>Q_V$.

IX. There cannot be two different functions, W_1 and W_2, which have equal values at every point of S_1 and S_2 (two closed surfaces the first of which shuts in the second), and between these surfaces are everywhere harmonic. If we suppose, for the sake of argument, that two such functions exist and call their difference u, it is clear that u is harmonic between the surfaces and that it vanishes at every point of both S_1 and S_2. If, therefore, in [145] we make $U=V=u$, we learn that

$$\int\int\int[(D_xu)^2+(D_yu)^2+(D_zu)^2]\,dx\,dy\,dz=0,$$

where the integral extends over all the space between S_1 and S_2. Since the integrand cannot be negative, it must be zero at every point, so that $D_xu=D_yu=D_zu=0$ and u is constant. But $u=0$ on S_1, therefore it is identically equal to zero and $W_1=W_2$.

It is easy to show that two functions which have equal normal derivatives at every point of S_1 and S_2, and are harmonic everywhere between the surfaces, can differ only by a constant.

X. We may now give an old proof of a theorem, originally discovered by Green from physical considerations, which is usually called Dirichlet's Principle by Continental writers, but in English books is generally attributed to Sir W. Thomson.[*] This theorem asserts that there always exists one, but no other than this one, function, v, of x, y, z, which (1) is continuous, and single-valued, together with its first space derivatives, throughout a given closed region T; (2) at every point of the region satisfies the equation $\nabla^2 v = 0$; and (3) at every point on the boundary of the region has any arbitrarily assigned value, provided that this can be regarded as the value at that point of a single-valued function, continuous all over this boundary.

There is evidently an infinite number of functions which satisfy the first and third conditions. If, for instance, the equation of the bounding surface S of the region is $F(x, y, z) = 0$, and if the value of v at the point (x, y, z) upon this surface is to be $f(x, y, z)$, any function of the form

$$\Phi(x, y, z) \cdot F(x, y, z) + f(x, y, z)$$

would satisfy the third condition, whatever continuous function Φ might be.

If we assign to the function to be found a constant value C all over S, $v \equiv C$ will satisfy all three of the conditions given above.

If the sought function is to have different values at different points of S, and if for u in the integral

$$Q = \int\int\int [(D_x u)^2 + (D_y u)^2 + (D_z u)^2]\, dx\, dy\, dz,$$

which is to be extended over the whole of the region, we substitute any one of all the functions which satisfy conditions (1) and (3), the resulting value of Q will be positive. Some one at least of these functions (v) must, however, yield a value of Q which, though positive, is so small that no other one can make Q smaller.[†] Let h be an arbitrary constant to which some value has been assigned, and let w be any function which satisfies condition (1) and is equal to zero at all parts of S, then $U = v + hw$ will satisfy conditions (1) and (3), and, conversely, there is no function which satisfies these two conditions which cannot be written in the form $U = v + hw$, where h is an arbitrary constant, and w some function which is zero at S and which satisfies condition (1).

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[*] W. Thomson, *Liouville's Journal*, 1847. *Dirichlet's Vorlesungen*, Bacharach, *Abriss der Geschichte der Potentialtheorie.*

[†] A principle which will doubtless lead to a justification of this by no means self-evident assumption was pointed out by Hilbert in a remarkable paper read before the Deutsche Mathematiker-Vereinigung in 1899.

Call the minimum value of Q due to v, Q_v, and the value of Q due to U, Q_U, then

$$Q_U = Q_v + 2h\int\int\int (D_x v \cdot D_x w + D_y v \cdot D_y w + D_z v \cdot D_z w)\, dx\, dy\, dz$$
$$+ h^2\int\int\int [(D_x w)^2 + (D_y w)^2 + (D_z w)^2]\, dx\, dy\, dz,$$

which, since w is zero at the boundary of the region, may be written, by the help of Green's Theorem,

$$Q_U - Q_v = -2h\int\int\int w\nabla^2 v\, dx\, dy\, dz + h^2\Omega^2.$$

Now, since Q_v is the minimum value of Q, no one of the infinite number of values of $Q_U - Q_v$ formed by changing h and w under the conditions just named can be negative; but if $\nabla^2 u$ were not everywhere equal to zero within T, it would be easy to choose w so that the coefficient of $2h$ in the expression for $Q_U - Q_v$ should not be zero, and then to choose h so that $Q_U - Q_v$ should be negative. Hence $\nabla^2 v$ is equal to zero throughout T, and there always exists at least one function which satisfies the three conditions stated above. Compare VIII.

There is only one such function; for if beside v there were another $u = v + hw$, we should have, since the coefficient of h is zero when $\nabla^2 v = 0$,

$$Q_u = Q_v + h^2\Omega^2,$$

and that Q_u may be as small as Q_v, $h\Omega$ must be zero, whence either $h = 0$ or $\Omega = 0$, and if $\Omega = 0$, w is zero. Therefore,

$u=v$, and there is only one function which in any given case satisfies all the three conditions given above.

XI. The potential function V, due to a volume distribution of finite density ρ in the region T and a superficial distribution of finite surface density σ on the surface S, is everywhere continuous, and it so vanishes at infinity that, if r is the distance from any finite point, each of the quantities

$$rV, \quad -r^2 D_r V,$$

as r becomes infinite, approaches the limit M, where M is the amount of matter (algebraically considered) in the whole distribution. The first derivatives of V are everywhere finite, and they are continuous except on S, at every point of which tangential derivatives are continuous, while the normal derivative is discontinuous in the manner indicated by the equation

$$D_{n_1} V + D_{n_2} V = -4\pi\sigma,$$

where n_1 and n_2 are the normals to the surface drawn away from it on each side. The second derivatives of V are everywhere finite, and they are continuous except at surfaces where ρ is discontinuous. At any point on such a surface the tangential second derivatives are continuous, but the normal second derivative is discontinuous by an amount equal to 4π times the discontinuity in ρ reckoned in the direction opposite to that in which the derivative is taken. Everywhere, except at surfaces of discontinuity in ρ, V satisfies Poisson's Equation, $\nabla^2 V = -4\pi\rho$, and without T, where there is no matter, this degenerates into Laplace's Equation.

For a given value of ρ in the given region T, and a given value of σ on the given surface S, only one function has all these properties. Assuming that there are two such functions, V and V', let their difference be the function u. At every point of S,

$$D_{n_1} V + D_{n_2} V = D_{n_1} V' + D_{n_2} V' = -4\pi\sigma,$$

so that
$$D_{n_1} u + D_{n_2} u = 0,$$

and even the normal derivatives of u are continuous at every point of S. At surfaces of discontinuity in ρ, the derivatives of u are all continuous and u satisfies everywhere Laplace's Equation. The limits, as r becomes infinite, of ru and $r^2 D_r u$ are zero. Since u with its first and second derivatives is everywhere continuous, we may imagine a spherical surface of large radius r, drawn about any finite point O, as centre, so as to enclose all the attracting mass and apply Green's Theorem in the form of [151] to u inside this surface. The numerical value of the surface integral

$$\int u\, D_n u\, dS$$

taken over the spherical surface is no greater than the area of the surface ($4\pi r^2$) multiplied by the largest value which $u \cdot D_r u$ has on the surface, or

$$4\pi\,[\text{greatest value of } (u\, r^2 D_r u)].$$

If, now, the radius of the surface be indefinitely increased, this expression approaches the limit zero so that the integral

$$\int\!\!\int\!\!\int [(D_x u)^2 + (D_y u)^2 + (D_z u)^2]\, dx\, dy\, dz$$

taken over all space has the value zero. Since the integrand is made up of squares which can never be negative, we must have at every point of space

$$D_x u = D_y u = D_z u = 0.$$

Therefore, u is constant in all space; and since it is zero at infinity, it must be everywhere zero, so that V and V' are identical. It is to be understood that T may be made up of several distinct regions, and that S may consist of several distinct surfaces.

49. The Surface Distributions Equivalent to Certain Volume Distributions. Keeping the notation of IV. in the last article, let S be a closed surface equipotential with respect either to the joint action of two distributions of matter, M_1 inside S and M_2 outside it, or (when M_2 equals zero) to the action of a single distribution within the surface; and let V_1, V_2, and V be the values of the potential functions due respectively to M_1 alone, to M_2 alone, and to M_1 and M_2 existing together. If a quantity of matter were condensed on S so as to give at every point a surface density equal to $\frac{-D_n V}{4\pi}$, the whole quantity of matter on the surface would be

$$\frac{-1}{4\pi}\int D_n V \cdot dS,$$

and this, by § 31, is equal in amount to M_1. Let us study the effect of removing M_1 from the inside of S and spreading it in a superficial distribution M_1' over S, so that the surface density at every point shall be $\frac{-D_n V}{4\pi}$. In what follows, it is assumed that we have two distributions of matter, one inside the closed surface and the other outside. It is to be carefully noted, however, that by putting M_2 equal to zero in our equations, all our results are applicable to the case where we have an equipotential surface surrounding all the matter, which may be all of one kind or not.

The value, at any point O, of the potential function due to the joint effect of M_2 and the surface distribution M_1', would be

$$V_o = V_2 - \frac{1}{4\pi}\int \frac{D_n V}{r} \cdot dS.$$

If O is an outside point, we have, by [153],

$$V_o = V_2 + V_1,$$

so that the effect at any point outside an equipotential surface of a quantity M_1 of matter anyhow distributed inside the surface is the same as that of an equal quantity of matter distributed over the surface in such a way that the superficial density at every point is $\frac{-D_n V}{4\pi}$, where V is the value of the potential function due to the joint action of M_1 and any matter (M_2) that may be outside the surface.

If O is an inside point, we have,

$$V_O = V_2 + V_s - V_2 = V_s,$$

which shows that the joint effect of M_2 and M_1' is to give to all points within and upon the surface the same constant value of the potential function which points upon the surface had before M_1 was displaced by M_1'. If, therefore, M_1' and M_2 exist without M_1, there is no force at any point of the cavity shut in by S; or, in other words, the force due to M_1' alone is at all points inside S equal and opposite to that due to M_2.

If M_1 and M_2 exist without M_1', the cavity enclosed by S is, in general, a field of force. M_1' acts as a screen to shield the space within S from the action of M_2.

The surface of M_1' is equipotential with respect to all the active matter, so that there is no tendency of the matter composing the surface distribution to arrange itself in any different manner upon S.

Since M_1' exerts the same force on every particle outside S that M_1 did, and since action and reaction are equal and opposite, every particle of M_2 exerts on M_1' forces the resultant of which is equal to the resultant of the forces with which the same particle urged M_1. The resultant effect, therefore, of the action of M_2 on M_1' is the same as the resultant effect of its action on M_1. Now the whole system of forces applied to the surface distribution by M_2 and by the repulsions for one another of its own parts is equivalent to a tension from without of $2\pi\sigma^2$ dynes per square centimeter applied all over S, and since the internal forces form a system in equilibrium, the resultant effect of M_2 on M_1 is equal to the resultant effect of the tension just mentioned on M_1'.

If two closed surfaces, S_1 and S_2, which mutually exclude each other, shut in, respectively, the two portions, M_1, M_2, of a distribution M, and are level surfaces of M's potential function, it is easy to see that a superficial distribution on S_1 of density $\sigma = -D_n V/4\pi$ would act on a particle without S_1 just as M_1 does, and that a similar distribution on S_2 would act on particles outside of S_2 as M_2 does. The action of M_1 on M_2 is the same as the resultant effect of the tension $2\pi\sigma^2$ or $(D_n V)^2/8\pi$ considered as acting all over S_2. The surface integral of $-D_n V/4\pi$ extended over any closed surface has been called by Maxwell the "electric displacement" through the surface.

50. Vectors. Stokes's Theorem. The Derivatives of Scalar Point Functions. It is frequently convenient to define a vector by giving the values (tensors) of its components parallel to the coördinate axes; and if for our present purposes we call these "the components of the vector," no confusion will arise. The expression (Q_x, Q_y, Q_z) denotes a vector, Q, the components of which parallel to the axes of x, y, and z are respectively equal to Q_x, Q_y, and Q_z. The direction cosines of the vector are the ratios of Q_x, Q_y, Q_z to $\sqrt{Q_x^2 + Q_y^2 + Q_z^2}$. The letter which represents a vector is often used in scalar equations to denote merely the tensor. Sometimes, however, the heavy face letter ($\boldsymbol{Q}$) is used to denote the vector, while its tensor is represented by the same letter in ordinary type. Any three scalar point functions can be considered the components of a vector point function. Scalar and vector point functions are sometimes called "distributed" scalars and vectors. Where there is no danger of any misunderstanding a vector point function may be called simply a vector.

The scalar function $D_x Q_x + D_y Q_y + D_z Q_z$ is called the *divergence* of Q, and if this quantity vanishes identically, Q is said to be a *solenoidal* vector. The force due to any finite distribution of matter attracting or repelling according to the "Law of Nature" is solenoidal in empty space. The negative of the divergence of a vector is called its *convergence*.

The vector, the components of which taken parallel to the coördinate axes are the three scalar point functions,

$$D_y Q_z - D_z Q_y, \; D_z Q_x - D_x Q_z, \; D_x Q_y - D_y Q_x,$$

is called the *curl* of Q; and if these components vanish at every point of a region, Q is said to be *lamellar* in that region. If the vector R is the curl of the vector Q, Q is said to be a *vector potential function* of R. The force due to a finite distribution of attracting or repelling matter is lamellar within and without the distribution. The curl of any vector is itself solenoidal. If two vectors have the same curl, their difference is a lamellar vector.

The *lines* of a vector are a family of curves, one of which passes through every point of space, and each of which has at every one of its points the direction of the vector at the point. The differential equations of the lines of the vector Q are evidently $$dx/Q_x = dy/Q_y = dz/Q_z:$$

after the values of Q_x, Q_y, and Q_z have been substituted, we have two equations of the form

$$\frac{dx}{dz} = \phi(x, y, z),\ \frac{dy}{dz} = \psi(x, y, z);$$

whence we get, by differentiating,

$$\frac{d^2x}{dz^2} = D_x\phi \cdot \frac{dx}{dz} + D_y\phi \cdot \psi + D_z\phi,$$

$$\frac{d^2y}{dz^2} = D_x\psi \cdot \phi + D_y\psi \cdot \frac{dy}{dz} + D_z\psi;$$

and, by eliminating y between the first and third equations, and x between the second and fourth equations, two equations of the second order between x and z and between y and z respectively. The integrals of these last equations are the equations of the lines of the vector. Sometimes the variables may be separated at the start, and then the work is much simplified. The lines of the vector $(-x^2, y, z)$ have the equations $y = Az$, $x \log(By) = 1$, and those of the vector $(3xz - yz, xz + yz, z)$, the equations $x = (B + A + Bz)\,e^{2z}$, $y = (A + Bz)\,e^{2z}$, where A and B are arbitrary constants.

If n represents the exterior normal of any closed surface S, the integral taken over S of the exterior normal component of the analytic vector Q is

$$\begin{aligned}\int Q \cos(n, Q)\,dS &= \int Q[\cos(x, Q)\cos(x, n) + \cos(y, Q)\cos(y, n) \\ &\qquad + \cos(z, Q)\cos(z, n)]\,dS \\ &= \int [Q_x \cos(x, n) + Q_y \cos(y, n) + Q_z \cos(z, n)]\,dS;\end{aligned}$$

and this is equal to the volume integral of the divergence of Q taken through the space within S. The integral of the exterior normal component of any analytic solenoidal vector. taken over any closed surface, is zero.

An important theorem due to Sir George Gabriel Stokes may be stated as follows:

The line integral taken around a closed curve s, of the tangential component of an analytic vector point function Q, is equal to the surface integral taken over any surface S, bounded by the curve, of the normal component of the curl of the vector, the direction of integration around the curve forming a right-handed screw rotation about the normals, or

$$\begin{aligned}\int [Q_x \cos(x, s) + Q_y \cos(y, s) + Q_z \cos(z, s)]\,ds \\ = \int\int [(D_y Q_z - D_z Q_y)\cos(x, n) \\ + (D_z Q_x - D_x Q_z)\cos(y, n) \\ + (D_x Q_y - D_y Q_x)\cos(z, n)]\,dS. \qquad [158]\end{aligned}$$

To prove this, we may evaluate first so much of the double integral as involves Q_x, that is,

$$\int\int [D_z Q_x \cdot \cos(y, n) - D_y Q_x \cdot \cos(z, n)]\,dS.$$

Let the area S be divided into quadrilateral elements by means of equally spaced planes parallel to the planes of zy and xy respectively, and consider especially one of these elements, ΔS, the projection of which on the xz plane is $\Delta x \cdot \Delta z$. so that $\Delta S \cdot \cos(y, n) = \Delta x \cdot \Delta z$ approximately.

That corner of the element ΔS which has the least x and z coördinates shall be the point P, and that side of the element which passes through P and is parallel to the plane of yz shall be represented by Δs_1. Since Δs_1 is perpendicular both to the normal to S at P and to the axis of x, $\cos(x, s_1) = 0$,

and
$$\cos(n, s_1) = \cos(x, n)\cdot\cos(x, s_1) + \cos(y, n)\cdot\cos(y, s_1) + \cos(z, n)\cdot\cos(z, s_1) = 0,$$

or
$$\frac{\cos(z, n)\cdot\cos(z, s_1)}{\cos(y, n)} = -\cos(y, s_1).$$

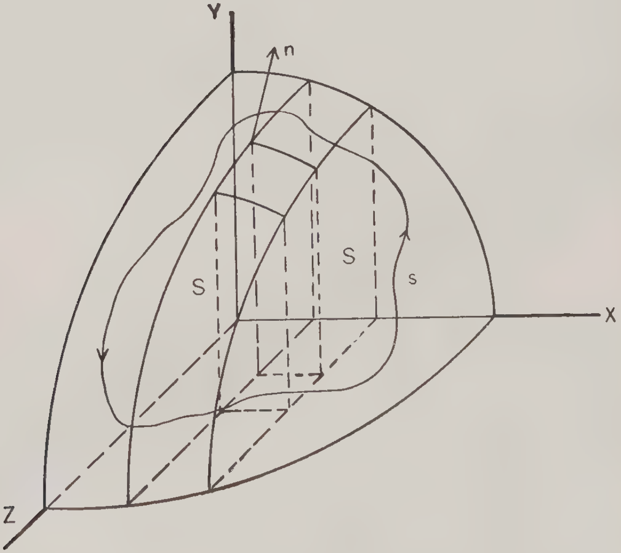

FIG. 38.

Moreover, $D_{s_1}Q_x = 0 + (D_yQ_x)\cos(y, s_1) + (D_zQ_x)\cos(z, s_1)$,

and
$$dS = \frac{dx\,dz}{\cos(y, n)} = \frac{dx\,dz\,ds_1}{ds_1\cos(y, n)} = ds_1\,dx\,\frac{\cos(z, s_1)}{\cos(y, n)}.$$

Hence, $\int\int [D_zQ_x \cdot \cos(y, n) - D_yQ_x \cdot \cos(z, n)]\,dS$

$$= \int\int \frac{[\cos(y,n) \cdot \cos(z,s_1)D_zQ_x - \cos(z,n)\cos(z,s_1)D_yQ_x]\,ds_1\,dx}{\cos(y, n)}$$

$$= \int\int [D_zQ_x \cdot \cos(z, s_1) + D_yQ_x \cdot \cos(y, s_1)]\,ds_1\,dx$$

$$= \int\int D_{s_1}Q_x \cdot ds_1\,dx.$$

If we perform the integration with respect to s_1 and introduce the limits, it will appear that this integral may be found by proceeding around the contour s in the direction indicated in the theorem and determining the line integral of

$$Q_x \frac{dx}{ds}\,ds = Q_x \cdot \cos(x, s)\,ds,$$

where ds is an element of s. If we treat in a similar manner those portions of the double integral which involve Q_y and Q_z, the theorem will be evident.

According to the definition used in the preceding sections, the numerical value of the *directional derivative* of any scalar point function u, at any point P, in any fixed direction PQ', is the limit, as PQ approaches zero, of the ratio of $u_Q - u_P$ to PQ, where Q is a point on the straight line PQ' between P and Q'. The *gradient* h_u of the function u at P is the directional derivative of u at P taken in the direction in which u increases most rapidly. This direction is normal to the surface of constant u which passes through P.

$$h_u^2 \equiv (D_xu)^2 + (D_yu)^2 + (D_zu)^2.$$

The directional derivative of any scalar point function at any point in any given direction is evidently equal to the product of the values of the gradient and the cosine of the angle between the given direction and that in which the function increases most rapidly.

The vector, the components of which parallel to the coördinate axes are numerically equal to D_xu, D_yu, D_zu, has been called the *vector differential parameter* of u. The numerical value (tensor) of this vector at any point is the gradient of u at the point, according to some writers; others use "gradient" to represent the vector itself. The lines of the vector are curves which cut orthogonally the surfaces of constant u, that is, the family of surfaces the equation of which is $u = c$, where c is a parameter constant for any one surface of the family.

If $f(x, y, z)$ is any scalar point function, any vector function the lines of which cut the surfaces of constant f normally must have components $R \cdot D_x f$, $R \cdot D_y f$, $R \cdot D_z f$, where R is some function of x, y, and z. The curl of this vector has components $D_z f \cdot D_y R - D_y f \cdot D_z R$, $D_x f \cdot D_z R - D_z f \cdot D_x R$, $D_y f \cdot D_x R - D_x f \cdot D_y R$, and the cosine of the angle between the vector and its curl is zero, so that these two vectors are perpendicular to each other. If a vector has a curl which is not perpendicular to it at every point, no family of surfaces exists the members of which cut the lines of the vector orthogonally at every point of space. Every plane vector point function has for its curl a vector perpendicular to its plane. The vector $(3\,yz, xz, xy)$ is not lamellar, but it is perpendicular to its curl: its lines cut orthogonally the family of surfaces $x^3yz = c$, as do the lines of the lamellar vector $(3\,x^2yz, x^3z, x^3y)$, each component of which is x^2 times the corresponding component of the first.

If the ratios of the corresponding components of two vector point functions are all equal to the same scalar point function, the vectors have the same lines. Two lamellar vectors may have the same lines, thus: the lines of every vector of the form $[f(x), 0, 0,]$ are parallel to the axis of x, and every such vector is lamellar, whatever analytic function f may represent.

We may define the numerical value of the *normal derivative* at any point P of a scalar point function u, taken with respect to another scalar point function v, to be the limit, as PQ approaches zero, of the ratio of $u_Q - u_P$ to $v_Q - v_P$, where Q is a point so chosen on the normal at P of the surface of constant v which passes through P that $v_Q - v_P$ is positive.

If (u, v) denotes the angle between the directions in which u and v increase most rapidly, the normal derivatives of u with respect to v, and of v with respect to u, may be written

$$h_u \cdot \cos(u, v) / h_v \text{ and } h_v \cdot \cos(u, v) / h_u$$

respectively. If $h_u = h_v$, these derivatives are equal.

The derivative of xyz with respect to $x + y + z$ has at the point $(1, 2, 3)$ the value $11/3$. The derivative at the same point of $x + y + z$ with respect to xyz is $11/49$.

51. The Attraction of Ellipsoids. If we transform the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

to parallel axes, using a point A_0, which lies on the surface and has the coördinates $(-x_0, -y_0, -z_0)$ as origin, and then denote by θ the angle which any radius vector drawn through A_0 makes with the x axis, the equation of the surface in polar coördinates takes the form

$$r'\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta\cos^2\phi}{b^2} + \frac{\sin^2\theta\sin^2\phi}{c^2}\right)$$
$$= 2\left(\frac{x_0\cos\theta}{a^2} + \frac{y_0\sin\theta\cos\phi}{b^2} + \frac{z_0\sin\theta\sin\phi}{c^2}\right).$$

If A_0 were at that extremity, A, of the a axis which has the coördinates $(-a, 0, 0)$, the equation would be

$$R\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta\cos^2\phi}{b^2} + \frac{\sin^2\theta\sin^2\phi}{c^2}\right) = \frac{2\cos\theta}{a}:$$

we will denote the coefficient of R in this equation by $\Omega(\theta, \phi)$.

Let us compare the x components of the attraction at A_0 and at A, due to a homogeneous ellipsoid of density ρ bounded by this surface. If, with each of these points as origin, a set of (conical) surfaces of constant θ with the constant difference $\Delta\theta$, and a set of (plane) surfaces of constant ϕ with the constant difference $\Delta\phi$, be imagined drawn, the ellipsoid will be divided into elementary "cones" in two ways. The vertices of all the cones of one system will be A, and the vertices of all the cones of the other system will be A_0. To every cone of the first system corresponds a cone with parallel axis belonging to the second system, but whereas every cone of the first system yields a positive contribution to the x force component at A, some of the corresponding cones of the second system yield negative components to the corresponding force component at A_0.

We shall find it convenient to write in parentheses after R and r' the value of θ and ϕ to which they belong, and to note that $r'_{(\pi-\theta,\,\pi+\phi)} = -r'_{(\theta,\,\phi)}$.

If the values of θ and ϕ which correspond to a given cone of the first system are θ_0 and ϕ_0, the values of θ and ϕ which belong to the corresponding cone of the second system may be either θ_0 and ϕ_0, or $\pi-\theta_0$ and $\pi+\phi_0$. The contribution of any cone of the first system to the x component of the force at A is

$$\rho\int_0^R r^2\sin\theta\, dr\, d\theta\, d\phi\cdot\frac{\cos\theta}{r^2} = \rho R_{(\theta,\,\phi)}\sin\theta\cos\theta\, d\theta\, d\phi,$$

and the contribution of the corresponding cone of the second system to the x component of the force at A_0 is either

$$\rho r'_{(\theta,\,\phi)}\sin\theta\cos\theta\, d\theta\, d\phi \quad\text{or}\quad \rho r'_{(\pi-\theta,\,\phi+\pi)}\sin\theta\cos\theta\, d\theta\, d\phi,$$

as the case may be.

If, now, we group together two cones of the first system corresponding to $(\theta_0,\ \phi_0)$ and $(\theta_0,\ \pi+\phi_0)$ respectively, we may write the positive contribution coming from this pair in the form

$$\rho\sin\theta_0\, d\theta\, d\phi\,\frac{4\cos^2\theta_0}{a\cdot\Omega(\theta_0,\ \phi_0)}.$$

The values of θ and ϕ for the corresponding cones of the second system are one of the pairs

$$(\theta_0,\ \phi_0;\ \theta_0,\ \pi+\phi_0),\ (\theta_0,\ \phi_0;\ \pi-\theta_0,\ \phi_0),$$
$$(\pi-\theta_0,\ \pi+\phi_0;\ \theta_0,\ \pi+\phi_0),\ \text{or}\ (\pi-\theta_0,\ \pi+\phi_0;\ \pi-\theta_0,\ \phi_0).$$

The two values of θ and ϕ of either of these pairs give equal and opposite values to

$$\left(\frac{y_0 \sin\theta \cos\phi}{b^2} + \frac{z_0 \sin\theta \sin\phi}{c^2}\right)\cos\theta,$$

so that the positive contributions of this pair of cones of the second system is

$$\rho \sin\theta_0 \, d\theta \, d\phi \, \frac{4\, x_0 \cos^2\theta_0}{a^2 \cdot \Omega\,(\theta_0, \phi_0)}.$$

This contribution to the x component of the force at A_0 is to the contribution of the corresponding cones of the first system to the corresponding force component at A as x_0 to a. Therefore, the x component at A_0 of the attraction due to the whole ellipsoid is to the corresponding component at A as x_0 to a.

If, then, we know the values (X_1, Y_1, Z_1) of the attraction due to a homogeneous ellipsoid bounded by the surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

at points on the surface at the negative extremities of the semiaxes a, b, c, we may find the numerical values of the components parallel to the coördinate axes of the attraction at any point $(-x_0, -y_0, -z_0)$ on the surface from the equations

$$X_0 = x_0 X_1/a, \quad Y_0 = y_0 Y_1/b, \quad Z_0 = z_0 Z_1/c.$$

The attraction X_1 at A can be easily found* by adding together the contributions coming from all the elementary cones with vertices at A into which the ellipsoid is divided, that is,

$$X_1 = \rho \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta \int_0^{2\pi} R_{(\theta,\,\phi)} \, d\phi, \text{ or, since}$$

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* See Routh's *Analytical Statics*, Vol. II, §§ 182–221. Tarleton's *Mathematical Theory of Attraction*, §§ 21–24 and 82–105. Schell's *Theorie der Bewegung und der Kräfte*, pp. 690–716.

$$R_{(\theta,\phi)} = \frac{2\,ab^2c^2\cos\theta}{b^2c^2\cos^2\theta + a^2c^2\sin^2\theta\cos^2\phi + a^2b^2\sin^2\theta\sin^2\phi}$$

$$= \frac{2\,ab^2c^2\cos\theta}{u + v\cos^2\phi + w\sin^2\phi},$$

$$X_1 = 8\,ab^2c^2\rho\int_0^{\pi/2}\sin\theta\cos^2\theta\,d\theta\int_0^{\pi/2}\frac{d\phi}{u+v\cos^2\phi+w\sin^2\phi}.$$

Now $\displaystyle\int_0^{\pi/2}\frac{d\phi}{u + v\cos^2\phi + w\sin^2\phi}$

$$= \int_0^{\infty}\frac{d\psi}{(u+v)+(u+w)\psi^2}, \text{ where } \psi = \tan\phi,$$

$$= \frac{\pi}{2\sqrt{(u+v)(u+w)}}.$$

Hence,

$$X_1 = 4\,abc\,\pi\rho\int_0^{\pi/2}\frac{\sin\theta\cos^2\theta\,d\theta}{\sqrt{(b^2\cos^2\theta + a^2\sin^2\theta)(c^2\cos^2\theta + a^2\sin^2\theta)}},$$

or, if $s = a^2\tan^2\theta$,

$$X_1 = 2\,a^2bc\,\pi\rho\int_0^{\infty}\frac{ds}{(s+a^2)^{3/2}(s+b^2)^{1/2}(s+c^2)^{1/2}},$$

and

$$X_0 = 2\,abc\,\pi\rho\,x_0\int_0^{\infty}\frac{ds}{(s+a^2)^{3/2}(s+b^2)^{1/2}(s+c^2)^{1/2}}$$

$$= 2\,abc\,\pi\rho\,x_0K_0 = x_0K_0', \qquad [159]$$

$$Y_0 = 2\,abc\,\pi\rho\,y_0\int_0^{\infty}\frac{ds}{(s+a^2)^{1/2}(s+b^2)^{3/2}(s+c^2)^{1/2}}$$

$$= 2\,abc\,\pi\rho\,y_0L_0 = y_0L_0',$$

$$Z_0 = 2\,abc\,\pi\rho\,z_0\int_0^{\infty}\frac{ds}{(s+a^2)^{1/2}(s+b^2)^{1/2}(s+c^2)^{3/2}}$$

$$= 2\,abc\,\pi\rho\,z_0M_0 = z_0M_0'.$$

At the positive ends of the axes of the ellipsoid the force components are $-X_1$, $-Y_1$, $-Z_1$. If the ellipsoid were made of matter of density ρ, *repelling* according to the "Law of Nature," the force components at the positive ends of the axes would be $+X_1$, $+Y_1$, $+Z_1$.

If (x_1, y_1, z_1) is a point on the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, $(\lambda x_1, \lambda y_1, \lambda z_1)$ is a *corresponding* point on the similar ellipsoid $\frac{x^2}{\lambda^2 a^2} + \frac{y^2}{\lambda^2 b^2} + \frac{z^2}{\lambda^2 c^2} = 1$, and the straight line which joins these two points passes through the origin.

It is to be noticed that K_0', L_0', M_0' have the same values for all similar ellipsoids, no matter what their actual dimensions may be, and that the components of the attraction at corresponding points on two similar homogeneous ellipsoids of equal density ρ are to each other as the linear dimensions of the ellipsoid.

Since the attraction of a homogeneous ellipsoidal homœoid is zero (Section 12) at all inside points, we may draw through any point P within a homogeneous ellipsoid bounded by a surface S_0, a surface S, concentric with S_0 and similar and similarly placed, and affirm that the attraction at P is equal to the attraction of so much of the whole ellipsoid as lies within S. If OP cuts S_0 in P_0, the attraction components at P are

$$X = -2\,abc\pi\rho\,xK_0,\quad Y = -2\,abc\pi\rho\,yL_0,\quad Z = -2\,abc\pi\rho\,zM_0,$$

or

$$-xK_0',\ -yL_0',\ -zM_0';$$

therefore, the resultant attractions at internal points on any straight line drawn through the centre of a homogeneous ellipsoid are parallel in direction. They are proportional in intensity to the distances of the points from the centre.

The potential function V within a homogeneous ellipsoid of density ρ bounded by the surface $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is such that its derivatives with respect to x, y, and z are respectively equal to $-2\,abc\pi\rho\,xK_0$, $-2\,abc\pi\rho\,yL_0$, $-2\,abc\pi\rho\,zM_0$, where K_0, L_0, M_0 have the same values at every point of the solid, so that

$$V = abc\pi\rho\,(G_0 - K_0x^2 - L_0y^2 - M_0z^2),$$

in which G_0 is a constant to be determined by computing $abc\pi\rho\,G_0$, the value of the potential function at the centre.

The polar equation of an ellipsoidal surface of semiaxes a, b, c, when the origin is at the centre and θ is the angle which any radius vector through the origin makes with the a axis, is

$$r_1^2 = \frac{a^2b^2c^2}{b^2c^2\cos^2\theta + a^2c^2\sin^2\theta\cos^2\phi + a^2b^2\sin^2\theta\sin^2\phi}$$

$$= \frac{a^2b^2c^2}{u + v\cos^2\phi + w\sin^2\phi}.$$

$$\text{Hence, } V_0 = G_0 abc\pi\rho = \rho\int_0^{\pi}\int_0^{2\pi}\int_0^{r_1} r\sin\theta\,d\theta\,d\phi\,dr$$

$$= 4\,a^2b^2c^2\rho\int_0^{\pi/2}\sin\theta\,d\theta\int_0^{\pi/2}\frac{d\phi}{u+v\cos^2\phi+w\sin^2\phi}.$$

Using the method of reduction already employed in finding the value of X_1, we learn that

$$G_0 = \int_0^{\infty}\frac{ds}{(s+a^2)^{1/2}(s+b^2)^{1/2}(s+c^2)^{1/2}}.$$

G_0 is an elliptic integral of the first kind, K_0, L_0, and M_0 are elliptical integrals of the second kind. If $a > b > c$,

$$(s + a^2) > (s + b^2) > (s + c^2) \text{ and } K_0 < L_0 < M_0,$$

and, unless s is zero,

$$(s + a^2)/(s + b^2) < a^2/b^2 \text{ and } (s + b^2)/(s + c^2) < b^2/c^2.$$

The equation for V may be written in the form

$$\frac{x^2}{\dfrac{V_0 - V}{K_0 abc\pi\rho}} + \frac{y^2}{\dfrac{V_0 - V}{L_0 abc\pi\rho}} + \frac{z^2}{\dfrac{V_0 - V}{M_0 abc\pi\rho}} = 1,$$

so that the equipotential surfaces within a homogeneous ellipsoid are a set of ellipsoidal surfaces coaxial with the given ellipsoid and similar to each other. The axes are in the same order of length as are those of the ellipsoidal mass, but are more nearly equal. The outer surface of the attracting ellipsoid is not equipotential.

The differential equations of the lines of force within a homogeneous ellipsoid are evidently

$$dx / xK_0 = dy / yL_0 = dz / zM_0,$$

so that if the reciprocals of K_0, L_0, M_0 are represented by k, l, m,

$$z^m = Cx^k = C_1 y^l.$$

The two ellipsoidal surfaces

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad \frac{x^2}{a'^2} + \frac{y^2}{b'^2} + \frac{z^2}{c'^2} = 1$$

are confocal if $a'^2 = a^2 + \lambda$, $b'^2 = b^2 + \lambda$, $c'^2 = c^2 + \lambda$. We will assume for convenience that λ is positive. A point P' on the second surface S' is said to *correspond* to a point P on the first surface S, if $x' : x = a' : a$, $y' : y = b' : b$, $z' : z = c' : c$.

If P_1 and P_2 are any two points on S, and P_1', P_2' the corresponding points on S', the distance P_1P_2' is equal to the distance $P_1'P_2$ [Ivory's Theorem], as may be seen by substituting for a', b', and c' in the following equation their values in terms of a, b, and c.

$$\left(x_1 - \frac{a'x_2}{a}\right)^2 + \left(y_1 - \frac{b'y_2}{b}\right)^2 + \left(z_1 - \frac{c'z_2}{c}\right)^2$$

$$= \left(x_2 - \frac{a'x_1}{a}\right)^2 + \left(y_2 - \frac{b'y_1}{b}\right)^2 + \left(z_2 - \frac{c'z_1}{c}\right).$$

To the points on a chord EF of S, drawn parallel to the x axis, correspond the points on a parallel chord $E'F'$ of S'. The lengths of these two chords are as a to a'. To the points in a slender prism Q, of cross-section $\Delta y \Delta z$, within S, one edge of which is the line EF, correspond the points in a slender prism Q', of cross-section $\Delta y' \Delta z'$, or $\Delta y \cdot \Delta z \cdot b'c' / bc$, within S', and one edge of this is the line $E'F'$.

If Q and Q' are made of homogeneous matter of equal density, the x component of the attraction at any point P', on the larger ellipsoid S', due to Q, is [Section 6] equal to

$$\rho \cdot \Delta y \cdot \Delta z \left(\frac{1}{P'E} - \frac{1}{P'F}\right),$$

and the x component of the attraction due to Q' at the point P on S corresponding to P' is

$$\frac{\rho\,\Delta y\,\Delta z\,b'c'}{bc}\left(\frac{1}{PE'}-\frac{1}{PF'}\right).$$

The quantities in the parentheses are equal, by Ivory's Theorem, and the two attraction components are to each other as $bc : b'c'$. If the whole space inside S' is filled with homogeneous matter of density ρ, the x component at any point P', on S', of the attraction of so much of the mass as lies within S is equal to the product of $\dfrac{bc}{b'c'}$ and the x component of the attraction of the whole mass at the inside point P which lies on S and corresponds to P'. We have already found an expression for the last-named force component.

To find, then, the attraction at the outside point $P'(x', y', z')$, due to a homogeneous ellipsoid of density ρ bounded by the surface S, or $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$, we must first find the positive value of λ which satisfies the cubic $\dfrac{x'^2}{a^2+\lambda}+\dfrac{y'^2}{b^2+\lambda}+\dfrac{z'^2}{c^2+\lambda}=1$, and thus determine the axes of the ellipsoidal surface S' through P' confocal with S. If we call this value of λ, λ', the point P on S which corresponds to P' on S' has the coördinates $\left(\dfrac{ax'}{\sqrt{a^2+\lambda'}},\ \dfrac{by'}{\sqrt{b^2+\lambda'}},\ \dfrac{cz'}{\sqrt{c^2+\lambda'}}\right)$, and the x component of the attraction at P due to an ellipsoid of density ρ bounded by S' would be

$$-\,2\,a'b'c'\pi\rho x\int_0^\infty \frac{ds}{(s+a'^2)^{3/2}(s+b'^2)^{1/2}(s+c'^2)^{1/2}}.$$

If we multiply this result by $bc/b'c'$, we shall get the result sought. If we substitute $s+\lambda$ for s in the integral and remember that $x : x' = a : a'$, we may write the x component of the attraction of the ellipsoid at the point P in the form

$$\begin{aligned} X &= -\,2\,abc\pi\rho x'\int_{\lambda'}^\infty \frac{ds}{(s+a^2)^{3/2}(s+b^2)^{1/2}(s+c^2)^{1/2}} \\ &= -\,2\,abc\pi\rho x' K = -\,\tfrac{3}{2}\,mx'K, \end{aligned} \qquad [160]$$

where m is the mass of the ellipsoid. The components parallel to the axes of y and z of the attraction at P' are, similarly,

$$Y = -\tfrac{3}{2}\, my' \int_{\lambda'}^{\infty} \frac{ds}{(s+a^2)^{1/2}(s+b^2)^{3/2}(s+c^2)^{1/2}}$$

$$= -2\, abc\pi\rho\, y' L = -\tfrac{3}{2}\, my' L,$$

$$Z = -\tfrac{3}{2}\, mz' \int_{\lambda'}^{\infty} \frac{ds}{(s+a^2)^{1/2}(s+b^2)^{1/2}(s+c)^{3/2}}$$

$$= -2\, abc\pi\rho\, z' M = -\tfrac{3}{2}\, mz' M.$$

We know that, if we substitute in the equation

$$F(\lambda) \equiv \frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} - 1 = 0,$$

the coördinates of any point in space, the largest root of the equation corresponds to an ellipsoid passing through the point, and is negative, zero, or positive according as the point lies within, on, or without S. Following Dirichlet, let us imagine a function u of the space coördinates, which shall have the value zero at every point within or on S, and, at every point outside of S, shall be equal to the positive root of the equation $F(\lambda) = 0$ which belongs to that point; and let us consider the integral

$$V = \pi abc\rho \int_{u}^{\infty} \left(1 - \frac{x^2}{s+a^2} - \frac{y^2}{s+b^2} - \frac{z^2}{s+c^2}\right) \frac{ds}{(s+a^2)^{1/2}(s+b^2)^{1/2}(s+c^2)^{1/2}},$$

which evidently vanishes at infinity. For inside points where u is zero, V is identical with the value just found for the potential function within a homogeneous ellipsoid of density ρ.

Since V involves x explicitly and also implicitly through u, we have, in general, at any outside point,

$$D_x V = -2\,\pi abc\rho\, x \int_{u}^{\infty} \frac{ds}{(s+a^2)^{3/2}(s+b^2)^{1/2}(s+c^2)^{1/2}}$$

$$-\frac{\pi abc\rho\, D_x u}{(u+a^2)^{1/2}(u+b^2)^{1/2}(u+c^2)^{1/2}}\left(1 - \frac{x^2}{a^2+u} - \frac{y^2}{b^2+u} - \frac{z^2}{c^2+u}\right);$$

but, from the definition of u, the coefficient of $D_x u$ vanishes when u is positive, so that the integral alone remains and gives the value already found for the x component of the attraction at an outside point due to a homogeneous ellipsoid of density ρ bounded by S. At S, $D_x V$ is continuous: V gives everywhere, therefore, the value of the potential function due to a homogeneous ellipsoid of density ρ bounded by S.

If we note that

$$\int\left(\frac{1}{s+a^2}+\frac{1}{s+b^2}+\frac{1}{s+c^2}\right)\frac{ds}{(s+a^2)^{1/2}(s+b^2)^{1/2}(s+c^2)^{1/2}}$$
$$=\frac{-2}{(s+a^2)^{1/2}(s+b^2)^{1/2}(s+c^2)^{1/2}},$$

and that the equation $F(u)=0$ yields

$$D_x u=\frac{2x}{(a^2+u)}\Big/\left(\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right)$$

(with similar values for $D_y u$ and $D_z u$), so that

$$\frac{x\,D_x u}{a^2+u}+\frac{y\,D_y u}{b^2+u}+\frac{z\,D_z u}{c^2+u}=2$$

for an outside point, and zero for a point within S, it is easy to see that V satisfies* Laplace's Equation without S and Poisson's Equation within S, as it should.

52. Logarithmic Potential Functions. When a distribution of matter attracting or repelling according to the "Law of Nature" is such that by a proper choice of axes of reference for a set of orthogonal Cartesian coördinates the density can be made to depend on two of these coördinates only, the distribution evidently extends indefinitely far in both directions parallel to the third axis. Such a distribution is sometimes said to be "columnar." Any infinitely long cylinder the density of every filament of which is the same throughout the whole length of that filament, though different filaments may have different densities, is a columnar distribution. If we choose for z axis a line parallel to these filaments, the components of the force taken parallel to the x and y axes at any point involve x and y only, and there is no force component parallel to the axis of z. Since the z coördinate will not appear in any of our equations, we may represent a columnar distribution by its trace in the xy plane, if we keep in mind the fact that the distribution itself extends to infinity in both directions perpendicular to this plane.

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* Picard, *Traité d'Analyse*, Vol. I, p. 177.

It is evident from the work of Section 6 that a fine, homogeneous filament of cross-section ΔA_1, made of repelling matter

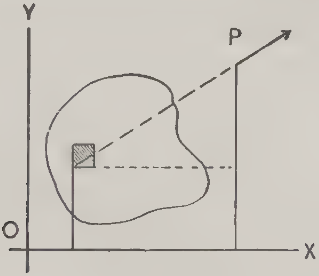

FIG. 39.

of density ρ_1, urges a unit mass at a point at a distance r from the filament with a force of $\frac{2\,\rho_1 \Delta A_1}{r}$ absolute kinetic force units. It follows that if the trace of a columnar distribution in the xy plane is an area A_1, the force components at the point (x, y, z) parallel to the axes of x and y are

$$X = \int\!\!\int \frac{2\,\rho_1 (x - x_1)\, dA_1}{(x_1 - x)^2 + (y_1 - y)^2}, \quad Y = \int\!\!\int \frac{2\,\rho_1 (y - y_1)\, dA_1}{(x_1 - x)^2 + (y_1 - y)^2},$$

where ρ_1 is the density at any point the x and y coördinates of which are x_1 and y_1 respectively, and where the integrals are to be extended over the whole of A_1. The integral

$$V = +\int\!\!\int \rho_1 \log\left[(x_1 - x)^2 + (y_1 - y)^2\right] dA_1 = \int\!\!\int 2\,\rho_1 \log r\, dA_1$$

extended over A is called from its form the "logarithmic potential function" belonging to the distribution, and

$$X = + D_x V, \quad Y = + D_y V.$$

In the general case the columnar distribution must be considered to be made up partly of filaments of positive matter and partly of filaments of negative matter, so that the density is positive for some values of x and y and negative for others. Under these circumstances X and Y represent the force components which would act on a unit quantity of positive matter concentrated at the point (x, y, z). It will be convenient to denote the amount of matter (reckoned algebraically) in the unit length of a columnar distribution by M. It is evident that at an infinite distance (in the xy plane) from the trace A_1 of a columnar distribution the logarithmic potential function becomes infinite, unless M is zero, while the force components vanish in any case.

It is easy to prove that, if M is zero, V so becomes infinite at infinity in the xy plane that, if r is the distance from any finite point in the plane, rV and $r^2 D_r V$ have finite limits as r increases indefinitely. If M is not zero, V becomes infinite at infinity in such a way that the quantities $(V - 2M\log r)$, $(r \cdot D_r V - 2M)$, $(r \cdot V \cdot D_r V - 4M^2 \log r)$, and $(V - r \cdot \log r \cdot D_r V)$ all approach the limit zero when r becomes infinite. That X, Y, and V are finite at every finite point in the xy plane outside of A_1 is evident; that no one of them is infinite at any point within A_1 can be proved by transforming the integrals which define them to polar coördinates, using the suspected point as origin.

If n is the exterior normal of any closed curve s in the xy plane, and r the distance from any fixed point O in the plane, the line integral of $\cos(n, r)/r$ taken around s is equal to zero, π, or 2π, according as O is without, on, or within s. From this it follows that the line integral around any closed curve in the xy plane, of the normal outward component of the force due to any columnar distribution of repelling matter the lines of which are perpendicular to that plane, is equal to 4π times the mass of the unit length of so much of the columnar distribution as is surrounded by the curve. We may regard this as Gauss's Theorem applied to columnar distributions.

If a function u involves x and y and does not involve z, no confusion need be caused by denoting $D_x^2u + D_y^2u$ by $\nabla^2 u$. Using this notation, Green's Theorem for functions of the two variables x and y may be written in the form

$$\int\int (D_xu \cdot D_xw + D_yu \cdot D_yw)\, dA$$

$$= \int u \cdot D_nw \cdot ds - \int\int u \cdot \nabla^2 w \cdot dA$$

$$= \int w \cdot D_nu \cdot ds - \int\int w \cdot \nabla^2 u \cdot dA,$$

where the line integrals are to be extended around a closed curve s in the xy plane, within and on which u and w with their first derivatives are continuous, and the double integrals extended over the area shut in by s. If in this equation we make $u = 1$ and w the logarithmic potential function V, due to a columnar distribution, we get

$$\int\int \nabla^2 V dA = \int D_n V ds,$$

and this, according to the special form of Gauss's Theorem, just stated, is equal to

$$\int\int 4\,\pi\rho\, dA.$$

Since the form of the curve s may be chosen at pleasure, it must be true that at every point $\nabla^2 V = +4\pi\rho$. It is desirable to notice that the plus sign here precedes $4\pi\rho$, whereas in Poisson's Equation, as applied to the Newtonian Potential Function of a finite mass, the corresponding sign is minus. This and many other differences of sign that appear in our equations might have been removed if the opposite sign had been given to the integral which defines the logarithmic potential of a columnar distribution, but if this had been done a positive mass would have given rise to a negative potential function, and this might have caused confusion.

If a portion of a columnar distribution consists of a surface charge on a cylindrical surface, we may conveniently construct a small quadrilateral in the xy plane by drawing two normals across the ends of an element of the trace of the cylindrical surface and two very near curves parallel to the trace element, one on one side and the other on the other. If, then, we apply Gauss's Theorem to this quadrilateral, we shall learn that at every point of the trace the sum of the normal derivatives of V taken away from the curve on each side is $4\pi\sigma$.

If a closed curve s be drawn in the xy plane so as to include the trace of a portion of a columnar distribution the lines of which are perpendicular to that plane and to exclude the trace of another portion, and if V_1 and V_2 represent the parts of the potential function V belonging to these two portions of the distribution, we may apply Green's Theorem to V and the logarithm of the distance from a fixed point O in the plane. If n represents a normal pointing outward from s, we shall find that

$$\int \frac{V \cdot \cos(n, r)}{r}\, ds - \int D_n V \cdot \log r \cdot ds$$

is equal to the value at O of $2\pi V_2$, if O is within s; and to the value at O of $-2\pi V_1$, if O is without s.

If s happens to be a curve on which V is constant,

$$\int 2\left(\frac{D_n V}{4\pi}\right) \log r \cdot ds$$

is equal to the value at O of V_1, if O is without s, or of $V_s - V_2$, if O is within s. The reader may compare these results with those given in equations [153] and [157].

If a function $w = f(x, y)$ has the value zero at every point of a closed curve s_1 in the xy plane and the constant value C all over another closed curve s_2, shut in by s_1, and if between s_1 and s_2, w is everywhere harmonic, we may apply Green's

Theorem to w and the logarithm of the distance from a fixed point O in the plane and prove that

$$\int 2\left(\frac{-D_n w}{4\pi}\right)\log r\cdot ds_1+\int 2\left(\frac{-D_n w}{4\pi}\right)\log r\cdot ds_2,$$

where the normals point outward on s_1 and inward on s_2, is equal to 0, the value of w at O, or C, according as O is without s_1, between s_1 and s_2, or within s_2. Surface charges, of density $\frac{-D_n w}{4\pi}$, applied to s_1 and s_2 would, therefore, give rise to the potential function w between s_1 and s_2.

If a function $w=f(x, y)$, harmonic at all finite points, has the constant value c on a closed curve s in the xy plane and becomes infinite at infinity in this plane in such a way that

$$\text{limit }(w-2\mu\log r)=0,\text{ or limit }(r\log r\cdot D_r w-w)=0,$$

where μ is a given constant, then at all points without s, if n is an interior normal,

$$w=\int 2\left(\frac{-D_n w}{4\pi}\right)\log r\cdot ds,$$

and w is the potential function due to a columnar distribution of superficial density $-D_n w/4\pi$ on the cylindrical surface of which s is the right section. The amount of matter in the unit length of this cylindrical distribution is μ.

If within the closed curve s in the xy plane, $w=f(x, y)$ is harmonic, we may apply Green's Theorem to w and the logarithm of the distance r from a fixed point O_1, within s, using as field the region within s and without a small circumference drawn around O_1. This yields

$$2\pi w_{\text{at } O_1}=\int[w\cdot D_n\log r_1-\log r_1\cdot D_n w]\,ds,$$

where n is the exterior normal to s. If r_2 is the distance from a fixed point O_2, without s, we may prove in a similar way that

$$0=\int[w\cdot D_n\log r_2-\log r_2\cdot D_n w]\,ds,$$

or $2\pi w_{\text{at } O_1} = \int [w(D_n \log r_1 - D_n \log r_2) + D_n w \cdot \log (r_2/r_1)]\, ds.$

If s is a circumference of radius a with centre at C, and if O_1 and O_2 are inverse points such that

$$CO_1 = l_1,\ \ CO_2 = l_2,\ \ l_1 l_2 = a^2,$$

then r_1/r_2 is constant all over s,

$$\int D_n w\, ds = \int\!\!\int \nabla^2 w\, dx\, dy = 0,$$

and
$$2\pi w_{\text{at } O_1} = \int w [D_n \log r_1 - D_n \log r_2]\, ds.$$

Moreover $r_1 \cdot D_n \log r_1 = \cos (r_1, n),\ \ r_2 \cdot D_n \log r_2 = \cos (r_2, n),$

$$l_1^2 = a^2 + r_1^2 - 2\, ar_1 \cos (r_1, n),$$
$$l_2^2 = a^2 + r_2^2 - 2\, ar_2 \cos (r_2, n),$$

and the value on s of r_1/r_2 is l_1/a, so that

$$w_{\text{at } O_1} = -\frac{1}{2\pi}\int \frac{w(l_1^2 - a^2)}{ar_1^2}\, ds$$

taken around the circumference.

If we introduce polar coördinates with origin at the centre of s and denote the coördinates of O_1 by l_1 and ϕ_1, we shall have

$$w_{\text{at } O_1} = -\frac{1}{2\pi}\int_0^{2\pi} \frac{w(l_1^2 - a^2)\, d\phi}{l_1^2 + a^2 - 2\, al_1 \cos (\phi - \phi_1)}. \quad [161]$$

This is sometimes called "Poisson's Integral."

At the centre of the circumference where $l_1 = 0$,

$$w = \frac{1}{2\pi a}\int w\, ds.$$

EXAMPLES.

1. If the potential function due to a certain distribution of matter is given equal to zero for all space external to a given closed surface S and equal to $\phi(x, y, z)$, where ϕ is a continuous single-valued function (zero at all points of S), in all space

within S; there is no matter without S, there is a superficial distribution of surface density

$$\sigma = \frac{1}{4\pi}[(D_x\phi)^2 + (D_y\phi)^2 + (D_z\phi)^2]^{\frac{1}{2}}$$

upon S, and the volume density of the matter within S is

$$\rho = -\frac{1}{4\pi}[D_x^2\phi + D_y^2\phi + D_z^2\phi].$$

[Thomson and Tait.]

2. Show that, if w is constant on the closed surface S and is harmonic within S, it is constant in the space enclosed by S; and that if W vanishes at infinity and is everywhere harmonic, it is everywhere equal to zero.

3. If two functions, w_1 and w_2, which without a closed surface S are harmonic and vanish at infinity, have on S values which at every point are in the ratio of λ to 1, λ being a constant, then everywhere $w_1 = \lambda w_2$.

4. The functions u and v have the constant values u_1 and v_1 on the closed surface S_1 and the constant values u_2 and v_2 on the closed surface S_2 within S_1. Between S_1 and S_2, u and v are harmonic. Show that

$$(u - u_1)(v_2 - v_1) = (v - v_1)(u_2 - u_1).$$

5. Outside a closed surface S, w_1 and w_2 are harmonic and have the same level surfaces. w_1 vanishes at infinity, while w_2 has everywhere at infinity the constant value C. Assuming that a scalar point function v is expressible in terms of another, u, if, and only if,

$$D_xv / D_xu = D_yv / D_yu = D_zv / D_zu,$$

show that w_2 is of the form $Bw_1 + C$.

6. Show that there cannot be two different functions, W and W', both of which within the space enclosed by a given surface S (1) satisfy Laplace's Equation, (2) are, together with their first space derivatives, continuous, and (3) are either equal at every point of S, or satisfy on S the equation $D_nW = D_nW'$, and are equal at some one point.

7. Show that, given a set of closed mutually exclusive surfaces, there cannot be two different functions, W and W', which without these surfaces (1) satisfy Laplace's Equation, (2) are, with their first space derivatives, continuous, (3) so vanish at infinity that rW, rW', r^2D_rW, r^2D_rW', where r is the distance from any finite fixed point, have finite limits, and which satisfy one of the following relations: (1) at every point on the given surfaces $W = W'$, (2) at every point of every surface $D_nW = D_nW'$.

8. At every point of a portion (or the whole) of a closed surface S (or of a set of closed surfaces) the functions w_1 and w_2 have equal values, and at every point of the remainder of S these functions have equal normal derivatives. Outside and on S both functions are harmonic, and they both vanish at infinity in some manner not more closely defined. Each of the integrals $\int D_nw_1\,dS$, $\int D_nw_2\,dS$ has evidently the same finite numerical value when taken over S or over any other surface which encloses S. Show that w_1 and w_2 are identical.

If the values of w_1 and w_2 at a point P, the coördinates of which referred to any fixed point as origin are (r, θ, ϕ), instead of approaching zero as r is made to increase indefinitely, both approach the limit $f(\theta, \phi)$, f being a continuous function, when, with any values of θ and ϕ, r is made infinite, w_1 and w_2 are identical.

9. The given closed surface S_1 shuts in the given closed surface S_2. The given function w is harmonic between S_1 and S_2. Show that no other function than w, harmonic between S_1 and S_2, has the same value that w has at every point of S_1 and the same value of the normal derivative at every point of S_2. Show also that any such function which has the same value of the normal derivative at every point of S_1 and S_2 that the normal derivative of w has differs from w at most by a constant. No other function than w_1, harmonic between S_1 and S_2, has the same value that w has at every point of S_2 and the same value of the normal derivative at every point of S_1.

10. The harmonic function w, which so vanishes at infinity that, if r is the distance from any fixed finite point, the limits of $r\,w$ and $r^2 D_r\,w$ are not infinite, has an open zero level surface S_1 as well as a series of closed level surfaces of which one is S_2. Show that in the region T, between S_1 and S_2, w is the potential function due to surface distributions on S_1 and S_2 defined by the equation $4\pi\sigma = D_n w$, where n points out of T. The whole charge on the two surfaces is zero.

11. Outside the closed surface S, upon which its value is given at every point, the function w is harmonic except at certain points, P_1, P_2, P_3, etc., where it becomes infinite in such a way that, if r_k represents the distance from P_k,

$$w - m_k / r_k \text{ is harmonic at } P_k,$$

where m_k is a constant belonging to the point P_k. At infinity w vanishes like a Newtonian potential function. Prove that w is unique. If w is a Newtonian potential function, what do you know about the distribution which gives rise to it?

12. The functions U, W, Θ, Ω with their first space derivatives are continuous, everywhere without a given closed surface S, and they vanish at infinity like a Newtonian potential function due to a finite distribution of matter. U and W have the same values at every point of S, but outside S, U, Θ, and Ω satisfy Laplace's Equation and W does not. The surface integrals of the normal derivatives of Θ and Ω taken over S are equal, but Θ has the same value all over S, and Ω a continuously variable value. Show that, if the integrations embrace all space outside S,

$$\iiint \{(D_x U)^2 + (D_y U)^2 + (D_z U)^2\}\,dx\,dy\,dz$$
$$< \iiint \{(D_x W)^2 + (D_y W)^2 + (D_z W)^2\}\,dx\,dy\,dz,$$

$$\iiint \{(D_x \Theta)^2 + (D_y \Theta)^2 + (D_z \Theta)^2\}\,dx\,dy\,dz$$
$$< \iiint \{(D_x \Omega)^2 + (D_y \Omega)^2 + (D_z \Omega)^2\}\,dx\,dy\,dz.$$

Hence, show that the energy of a given charge spread on a given surface S is least when the arrangement is equipotential.

13. Everywhere within the closed surface S the two scalar point functions V and V' are continuous with their first derivatives. Over a given portion of S, V and V' have equal values, while over the remainder of S both $D_n V$ and $D_n V'$ are equal to zero. The vectors q and q' have the components $\lambda D_x V$, $\lambda D_y V$, $\lambda D_z V$ and $\lambda D_x V'$, $\lambda D_y V'$, $\lambda D_z V'$ respectively, where λ is a positive analytic scalar point function. Show that, if q is solenoidal and q' is not solenoidal, the integral

$$\iiint \lambda[(D_x V)^2 + (D_y V)^2 + (D_z V)^2]\,d\tau$$

extended over the whole space within S is less than the integral

$$\iiint \lambda[(D_x V')^2 + (D_y V')^2 + (D_z V')^2]\,d\tau$$

extended over the same region.

14. Gravitating matter of given uniform density is confined within a given closed surface, but its volume is less than that enclosed by the surface. Prove that its potential energy is a maximum, if the matter forms a shell of which the given surface is the outer boundary, while the internal boundary is an equipotential surface.

15. Let $\xi \equiv f_1(x, y)$ and $\eta \equiv f_2(x, y)$ be two analytical functions of x and y such that the two families of curves

$$f_1(x, y) = c, \quad f_2(x, y) = k$$

are orthogonal. Let V be any function of x and y which, with its first space derivatives, is continuous, within and on a closed curve s, drawn in the coördinate plane. Let h_ξ and h_η be the positive roots of the equations

$$h_\xi^2 = (D_x \xi)^2 + (D_y \xi)^2, \quad h_\eta^2 = (D_x \eta)^2 + (D_y \eta)^2.$$

Prove that s, the surface integral of $h_\xi \cdot h_\eta \cdot D_\xi\left(\dfrac{V}{h_\eta}\right)$, taken all over the area enclosed by s, is equal to the line integral taken around s of $V\cos(\xi, n)$, where n is an exterior normal and (ξ, n) represents the angle between n and the direction in which ξ increased most rapidly.

Show that the corresponding theorem in three dimensions may be expressed by the equation

$$\int\int\int h_\xi h_\eta h_\zeta \cdot D_\xi\left(\frac{V}{h_\eta h_\zeta}\right) d\tau = \int\int V\cos(\xi, n)\, dS.$$

16. The operator $[(D_x)^2 + (D_y)^2 + (D_z)^2]$ applied to any of the quantities $x \pm y \pm iz\sqrt{2}$, $x \pm iy\sqrt{2} \pm z$, etc., yields zero: is every analytic function of any one of these quantities harmonic?

17. The product of two harmonic functions, u, v, is itself harmonic if, and only if, the level surfaces of u and v are orthogonal. The product of three harmonic functions, u, v, w, is itself harmonic if, and only if, the level surfaces of u, v, and w are mutually orthogonal.

18. The function w of the two variables x and y is harmonic in the xy plane everywhere outside of the mutually exclusive closed curves s_1 and s_2. Upon these curves w has given constant values. At infinity, w becomes infinite in such a manner that, if r is the distance from any finite point in the xy plane,

$$\underset{r=\infty}{\text{limit}} (r \log r \cdot D_r w - w) = 0.$$

Show that w is the potential function without s_1 and s_2, due to superficial distributions defined by the equation $4\pi\sigma = D_n w$, upon the cylindrical surfaces of which s_1 and s_2 are the traces. In the formula just given the normal points outward at s_1 and s_2.

19. The function w of the two variables x and y is harmonic everywhere in the xy plane except at certain points, P_1, P_2, P_3, etc., where it becomes infinite in such a manner that, if r_k is the distance from P_k, $w - 2\mu_k \log r_k$ is harmonic at P_k where μ_k is a constant belonging to P_k. Upon a certain open

curve, s, w has the value zero, and everywhere at an infinite distance from the origin w so becomes infinite that

$$\underset{r=\infty}{\text{limit}} (r \cdot \log r \cdot D_r w - w) = 0:$$

show that on either side of s, w may be considered as the logarithmic potential function due to a distribution of electricity of density $\sigma = -\dfrac{D_r w}{4\pi}$ on the infinite cylindrical surface of which s is a right section, and to distributions upon lines normal to the xy plane which cut the plane at so many of the P points as lie on the chosen side of s.

20. If the normal component of a vector is zero at every point of a closed surface S, and if within and on S the vector is everywhere solenoidal and lamellar, its components are equal to zero at every point within S. If the normal component of a vector is given at every point of S, and if everywhere within S the curl and the divergence have given values, the vector is determined. If q and q' are vectors the normal components of which vanish at every point of S, and if within S, q is solenoidal with curl k, while q' is lamellar with divergence D, where k and D are given scalar point functions, $q + q'$ is the unique vector, the normal component of which is zero at every point of S, and which within S has the curl k and the divergence D.

21. The normal derivative of u with respect to v is

$$(D_x u \cdot D_x v + D_y u \cdot D_y v + D_z u \cdot D_z v)/h_v^2.$$

22. If $u \equiv xyz$, $v \equiv 2x + y + z$, the values at $(1, 1, 1)$ of $D_v u$ and $D_u v$ are $2/3$ and $4/3$.

23. The gradients of u and v are numerically equal at every point, though not in general coincident in direction, if, and only if, $u + v$ and $u - v$ are orthogonal functions. If the gradients of u and v agree everywhere in direction though not in magnitude, v is expressible as a function of u, so that $v = f(u)$, $h_v = f'(u) \cdot h_u$.

24. If the components parallel to the axes of x and y of the solenoidal vector $(u, v, 0)$, which has no component parallel to the z axis, are independent of z, a vector, directed parallel to the z axis, which has for its intensity any partial integral (Q_z) of u with respect to y which satisfies the condition $D_x Q_z = -v$, is a vector potential function of the original vector. Thus: $(0, 0, x^2 y + y^3 - x^9)$ is a vector potential function of $(x^2 + 3y^2, 9x^8 - 2xy, 0)$. The value at the point (x, y, z) of the derivative of Q_z, taken in a direction perpendicular to the z axis and making an angle $\alpha + 90°$ with the plane of xz, is $D_x Q_z \cdot \cos(\alpha + 90°) + D_y Q_z \cdot \sin(\alpha + 90°)$, or $D_y Q_z \cdot \cos\alpha - D_x Q_z \cdot \sin\alpha$, or $u \cos\alpha + v \sin\alpha$, and this is the resolved part of the vector $(u, v, 0)$ at the same point in a direction parallel to the xy plane and making an angle α with the plane of xz. We learn, therefore, that the numerical value at any point P of the derivative of Q_z, taken in any direction s parallel to the xy plane, is equal to the component of the vector $(u, v, 0)$ in a direction parallel to the xy plane and perpendicular to s. Show that the intersection of any plane parallel to the xy plane with a cylinder of the family $Q_z = \text{constant}$ is a line of the vector $(u, v, 0)$. Show also that $D_x^2 Q_z + D_y^2 Q_z = -(D_x v - D_y u)$, the negative of the component parallel to the z axis of the curl of $(u, v, 0)$.

25. A vector parallel to the x axis of intensity independent of z and equal to the negative of a partial integral of w with respect to y, and a vector parallel to the y axis of intensity independent of z and equal to a partial integral of w with respect to x, are vector potential functions of the vector $(0, 0, w)$, provided w is independent of z. For example: the vectors $[y^2 - 3x^2 y + f(x), 0, 0]$ and $[0, x^3 - 2xy + \phi(y), 0]$ are vector potential functions of the vector $(0, 0, 3x^2 - 2y)$.

26. If the lines of a vector are circles parallel to the xy plane with centres on the z axis, and if the intensity of the vector is a function $f(r)$ of the distance r from that axis, a vector, everywhere parallel to the z axis, of intensity $F(r)$,

where $f(r) = -D_r F(r)$ is a vector potential function of the original vector. Is this original vector solenoidal?

27. If the lines of a vector are straight lines parallel to the xy plane and emanating from the z axis, and if the intensity of the vector is a function $f(r)$ of the distance r from this axis, $f(r)$ must be of the form c/r if the vector is solenoidal. A vector with such lines as these cannot be solenoidal if the intensity at every point is a given function of the angle which the line of the vector through that point makes with the xz plane.

28. The lines of the vector $[x \cdot f(x, y),\ y \cdot f(x, y),\ 0]$ are straight lines parallel to the xy plane and emanating from the z axis, and its curl is of the form $(0, 0, y \cdot D_x f - x \cdot D_y f)$. If f is expressible as a function of the angle $\tan^{-1}(y/x)$, $y \cdot D_x f - x \cdot D_y f$ is also expressible as a function of this angle, but if f is expressible as a function of $r \equiv \sqrt{x^2 + y^2}$, $y \cdot D_x f - x \cdot D_y f$ vanishes and no vector of the form

$$[x \cdot f(r), y \cdot f(r), 0]$$

can be a vector potential function of the vector $[0, 0, \phi(r)]$.

If the ratio of y to x be denoted by μ, and if $f(\mu) \equiv -\int \frac{\phi(\mu)\, d\mu}{\mu^2 + 1}$, the vector $[x \cdot f(\mu),\ y \cdot f(\mu),\ 0]$ is a vector potential function of the vector $[0, 0, \phi(\mu)]$.

29. The lines of the vector $[-y \cdot f(x, y),\ x \cdot f(x, y),\ 0]$ are circles parallel to the xy plane with centres on the z axis, and its curl is of the form $(0, 0, 2f + x \cdot D_x f + y \cdot D_y f)$. Show that if f is expressible as a function of r, the distance from the z axis, so is $2f + x \cdot D_x f + y \cdot D_y f$, and that, if

$$F(r) \equiv \frac{1}{r^2} \int r \cdot \phi(r)\, dr,$$

$[-y \cdot F(r),\ x \cdot F(r),\ 0]$ is a vector potential function of the solenoidal vector $[0, 0, \phi(r)]$. Show also that if f is expressible as a function of the angle $\tan^{-1}(y/x)$, that is, as a function of the ratio, μ, of y to x, $2f + x \cdot D_x f + y \cdot D_y f$ is expressible as a function of μ, and that $[-\frac{1}{2}y \cdot f(\mu), \frac{1}{2}x \cdot f(\mu), 0]$ is a vector potential function of $[0, 0, f(\mu)]$.

30. The difference between the values at any two points A and B of any analytic scalar point function V is equal to the line integral taken along any path from A to B of the tangential component of the vector $(D_x V, D_y V, D_z V)$.

31. The only families of plane curves which are at once the right sections of possible systems of equipotential cylindrical surfaces in empty space due to columnar distributions of matter which attracts according to the "Law of Nature," and also the generating curves of possible systems of equipotential surfaces of revolution due to distributions of such matter symmetrical about the common axis of these surfaces, are families of concentric conics. Must every such family of conics be confocal? [*Am. Jour. Math.*, 1896.]

32. If a vector is determined at every point by means of the components (R, Θ, Z) in the directions in which the columnar coördinates of the point increase most rapidly, the divergence of the vector may be written $D_r R + R/r + D_\theta \Theta / r + D_z Z$.

33. The equation

$$F(\lambda) \equiv \frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} - 1 = 0$$

represents, when a, b, and c are fixed, a family of confocal quadric surfaces of which λ is the parameter. If $a > b > c$, and if x, y, and z are chosen at pleasure, the cubic in λ has three real roots (u, v, w); one between $-a^2$ and $-b^2$, corresponding to a parted hyperboloid, one between $-b^2$ and $-c^2$, corresponding to an unparted hyperboloid, and one between $-c^2$ and ∞, corresponding to an ellipsoid, so that through every point of space three surfaces of the family can be drawn, and it is easy to see that these cut each other orthogonally. The direction cosines of a surface of constant λ have the values $D_x\lambda / h$, $D_y\lambda / h$, $D_z\lambda / h$, where $h^2 = (D_x\lambda)^2 + (D_y\lambda)^2 + (D_z\lambda)^2$.

$$D_x\lambda = -2x/(a^2+\lambda)D_\lambda F, \text{ and } h^2 = -4/D_\lambda F.$$

Belonging to every point in space are three values of λ (u, v, w), and three values of h (h_u, h_v, h_w), and, if we substitute u, v, and w successively for λ in the equation $F(\lambda)=0$, we shall get three linear equations in x^2, y^2, z^2 from which we may obtain expressions for x, y, z in terms of u, v, w.

$$h_u^2 = -4[(a^2+u)(b^2+u)(c^2+u)]/[(u-v)(u-w)],$$

and h_v^2 and h_w^2 have corresponding values which, substituted in

$$D_u\left\{\frac{h_u}{h_v h_w} D_u V\right\} + D_v\left\{\frac{h_v}{h_u h_w} D_v V\right\} + D_w\left\{\frac{h_w}{h_u h_v} D_w V\right\} = 0,$$

gives Laplace's Equation in terms of the orthogonal curvilinear coördinates (u, v, w). Prove that if we assume that a solution of this equation exists which involves w only and vanishes when w is infinite, the equation which determines this solution takes the form

$$D_w\{[(a^2+w)(b^2+w)(c^2+w)]^{1/2}\cdot D_w V\} = 0,$$

so that
$$V = C' - C\int \frac{dw}{(a^2+w)^{1/2}(b^2+w)^{1/2}(c^2+w)^{1/2}}$$

$$= C\int_w^\infty \frac{dw}{(a^2+w)^{1/2}(b^2+w)^{1/2}(c^2+w)^{1/2}}.$$

Hence, show that a set of confocal ellipsoids are possible external equipotential surfaces, and that if M is the mass of the corresponding distribution the potential function is given by the last equation, in which, since a very large value of w corresponds to an ellipsoid little different from a sphere of radius $\sqrt{w}$, C is to be determined by the equation

$$\underset{w=\infty}{\text{limit}}\; V\sqrt{w} = M.$$

Find the density of a superficial distribution on a surface of the w family, the potential function of which at all outside points shall be the function just defined.

34. The curl of the curl of a solenoidal vector such that the three functions which give the strengths of its components parallel to the coördinate axes satisfy Laplace's Equation vanishes. If the lines of a vector are all parallel to a plane, and the vector has the same value at all points in any line perpendicular to the plane, the vector is perpendicular to its curl.

35. A certain vector, the tensor of which is $f(x, y, z)$, is at every point directed exactly in the direction of the straight line which joins the origin with the point in question; show that the vector is not necessarily lamellar, but that it is perpendicular to its curl. If all the components of a vector are functions of x and y only, or if all are functions of x only, or if one component vanishes and the other components are functions of x, y, and z, the vector may or may not be perpendicular to its curl.

36. If (Q_x, Q_y, Q_z) are the components of a vector Q, $(\lambda_1, \mu_1, \nu_1)$ the curl components, $(\lambda_2, \mu_2, \nu_2)$ the components of the curl of the curl of Q, and so on,

$$\lambda_1 = D_y Q_z - D_z Q_y, \qquad \lambda_2 = D_x (\text{Div } Q) - \nabla^2 Q_x,$$

$$\lambda_3 = -\nabla^2 \lambda_1, \qquad \lambda_4 = -\nabla^2 \lambda_2, \text{ and so on.}$$

How are these equations changed if Q is a solenoidal vector?

37. If the harmonic function $f(x, y, z)$ represents the x component of a vector which is both solenoidal and lamellar, the y and z components must be of the form

$$Y = \int D_y f \cdot dx + D_y \psi(y, z), \quad Z = \int D_z f \cdot dx + D_z \psi(y, z),$$

where $\psi(y, z)$ is a solution of the equation

$$D_y^2 \psi + D_z^2 \psi = - D_x f.$$

38. A certain vector (X, Y, Z) is not perpendicular to its curl (K_x, K_y, K_z). Show that the scalar function F, determined from the equation

$$K_x \cdot D_x F + K_y \cdot D_y F + K_z \cdot D_z F = -(K_x X + K_y Y + K_z Z),$$

is the scalar potential function of a lamellar vector (L, M, N), which added to the first vector gives a new vector perpendicular to its curl. Is this equation always integrable?

39. A vector Q, with components (Q_x, Q_y, Q_z), is continuous except at a certain surface S. In each of the regions separated by S, $D_xQ_y = D_yQ_x$, $D_xQ_z = D_zQ_x$, $D_yQ_z = D_zQ_y$, so that at every point within these regions the curl of Q vanishes. Investigate the value of the curl of Q *on* S when the normal (or a tangential) component of Q is discontinuous there.

40. Unless $\nabla^2 f = 0$, a vector the x component of which is $f(x, y, z)$ cannot be both lamellar and solenoidal.

41. Matter spread uniformly in a superficial distribution on a circular portion of a plane forms a "circular surface distribution." Two such distributions, each of radius a, are placed parallel and opposite each other at a distance δ apart. If the density of one of these be $+\sigma$ and that of the other $-\sigma$, and if δ be made to approach zero and σ to increase in such a manner that the product of σ and δ is always equal to the constant μ, the resulting value of the potential function is said to be due to a "circular double layer" of radius a, and density μ. Show that the limiting value of the potential function at a point P on the axis of the double layer and at a distance x from its plane is $\pm 2\pi\mu(1 - x/\sqrt{a^2 + x^2})$, where the positive sign is to be used if P is on one side of the double layer, and the negative sign if P is on the other side. Is the potential function discontinuous at the double layer? Is the force discontinuous?

42. Assuming the surface of the earth as defined by the sea-level to be a spheroid of ellipticity ϵ, prove that the mass of the earth in astronomical units is $a_0{}^2 g_0 (1 + \epsilon - \frac{3}{2} m)$, where g_0 is the force of gravity at the equator, a_0 the equatorial radius, and m the ratio of "centrifugal force" to true gravity at the equator.

CHAPTER V.

THE ELEMENTS OF THE MATHEMATICAL THEORY OF ELECTRICITY.

I. ELECTROSTATICS.

53. Introductory. Having considered abstractly a few of the characteristic properties of what has been called "the Newtonian potential function," we will devote this chapter to a very brief discussion of some general principles of Electrostatics and Electrokinetics. By so doing we shall incidentally learn how to apply to the treatment of certain practical problems many of the theorems that we have proved in the preceding chapters.

In what follows, the reader is supposed to be familiar with such electrostatic phenomena as are described in the first few chapters of treatises on Statical Electricity, and with the hypotheses that are given to explain these phenomena.

Without expressing any opinion with regard to the physical nature of what is called *electrification*, we shall here take for granted that whether it is due to the presence of some substance, or is only the consequence of a mode of motion or of a state of polarization, we may, without error in our results, use some of the language of the old "Two Fluid Theory of Electricity" as the basis of our mathematical work.

The reader is reminded that, among other things, this theory teaches that: —

(1) Every particle of a body which is in its natural state contains, combined together so as to cancel each other's effects at all outside points, equal large quantities of two kinds of *electricity* with properties like those of the positive and negative "matter" described in Section 44.

(2) Electrification consists in destroying in some way the equality between the amounts of the two kinds of electricity which a body, or some part of a body, naturally contains, so that there shall be an excess or *charge* of one kind. If the

charge is of positive electricity, the body is said to be positively electrified; if the charge is negative, negatively electrified. Either kind of electricity existing uncombined with an equal quantity of the other kind, is called *free* electricity.

(3) When a charged body A is brought into the neighborhood of another body B in its natural state, the two kinds of electricity in every particle of B tend to separate from each other, one being attracted and the other repelled by A's charge, and to move in opposite directions.

In general, a tendency to separation occurs in all parts of the body, whether it is charged or not, where the resultant electric force (the force due to all the free electricity in existence) is not zero. This effect is said to be due to *induction*.

In our work we shall assume all this to be true, and proceed to apply the principles stated in Section 44 to the treatment of problems involving distributions of electricity. We shall find it convenient to distinguish between *conductors*, which offer practically no resistance to the passage of electricity through their substance, and *nonconductors*, which we shall regard as preventing altogether such transfer of electricity from part to part.

54. The Charges on Conductors are Superficial. When electricity is communicated to a conductor, a state of equilibrium is soon established. After this has taken place, there can be no resultant force tending to move any portion of the charge through the substance of the conductor, for, by supposition, the conductor does not prevent the passage of electricity through itself.

Moreover, the resultant electric force must be zero at all points in the substance of a conductor in electric equilibrium; for if the force were not zero at any point, electricity would be produced by induction at that point, and carried away through the body of the conductor under the action of the inducing force.

From this it follows that the potential function V, due to all the free electricity in existence, must be constant throughout the substance of any single conductor in electric equilibrium, whether or not the conductor be charged, and whether or not there be other charged or uncharged conductors in the neighborhood. Different conductors existing together will in general be at different potentials, but all the points of any one of these conductors will be at the same potential.

Wherever V is constant, $\nabla^2 V = 0$, and hence, by Poisson's Equation, $\rho = 0$, so that there can be no free electricity within the substance of a conductor in equilibrium, and the whole charge must be distributed upon the surface. Experiment shows that we must regard the thickness of charges spread upon conductors as inappreciable, and that it is best to consider that in such cases we have to do with really superficial distributions of electricity, in which the conductor bears a rough analogy to the cavity enclosed by the thin shells of repelling matter described in the preceding chapter.

The surface density at any point of a superficial distribution of electricity shall be taken positive or negative, according as the electricity at that point is positive or negative, and the force which would act upon a unit of positive electricity if it were concentrated at a point P without disturbing existing distributions shall be called "the electric force" or "the strength of the electric field at P."

It is evident, from Sections 45 and 46, that the electric force at a point just outside a charged conductor, at a place where the surface density of the charge is σ, is $4\pi\sigma$, and that this is directed outwards or inwards, according as σ is positive or negative.

In other words, $D_n V$, the derivative of the potential function in the direction of the exterior normal, is equal to $-4\pi\sigma$, and the value of V at a point P just outside the conductor is greater or less than its value within the conductor, according as the surface density of the conductor's charge in the neighborhood of P is negative or positive.

It is to be carefully noted that, although the surface of a conductor must always be equipotential, the superficial density of the conductor's charge need not be the same at all parts of the surface. We shall soon meet with cases where the electricity on a conductor's surface is at some points positive and at others negative, and with other cases where the sign of the potential function inside and on a conductor is of opposite sign to the charge.

It is evident, from the work of Section 47, that the resistance per unit of area which the nonconducting medium about a conductor has to exert upon the conductor's charge to prevent it from flying off, is, at a part where the density is σ, $2\pi\sigma^2$.

55. General Principles which follow directly from the Theory of the Newtonian Potential Function. If two different distributions of electricity, which have the same system of equipotential surfaces throughout a certain region, be superposed so as to exist together, the new distribution will have the same equipotential surfaces in that region as each of the components. For, if V_1 and V_2, the potential functions due to the two components respectively, be both constant over any surface, their sum will be constant over the same surface.

Two distributions of electricity, which have densities everywhere equal in magnitude but opposite in sign, have the same system of equipotential surfaces, and, if superposed, have no effect at any point in space.

Two distributions of electricity, arranged successively on the same conductor so that at every point the density of the one is m times that of the other, have the same system of equipotential surfaces, and the potential function due to the first is everywhere m times as great as that due to the second.

If the whole charge of a conductor which is not exposed to the action of any electricity except its own is zero, the superficial density must be zero at all points of the surface, and the conductor is in its natural state. For if σ is not everywhere zero, it must be in some places positive and in others negative; and, according to the work of the last section, the potential function V, due to this charge, must have, somewhere outside the conductor, values higher and lower than V_0, its value in the conductor itself. But this would necessitate somewhere in empty space a value of the potential function not lying between V_0 and 0, the value at infinity; that is, a maximum in empty space if V_0 is positive, and a minimum if V_0 is negative; which is absurd.

A system of conductors, on each of which the charge is null, must be in the natural state if exposed to the action of no outside electricity. For, by applying the reasoning just used to that conductor in which the potential function is supposed to have the value most widely different from zero, we may show that the surface density all over the conductor is zero, so that no influence is exercised on outside bodies; and then, supposing this conductor removed, we may proceed in the same way with the system made up of the remaining conductors.

If a charge M of electricity, when given to a conductor, arranges itself in equilibrium so as to give the surface density $\sigma = f(x, y, z)$ and to make the potential function $V_0 = \int \frac{\sigma\, ds}{r}$ constant within the conductor, a charge $-M$, if arranged on the conductor so as to give at every point the density $-\sigma = -f(x,y,z)$ would be in equilibrium, for it would give everywhere the potential function $\int \frac{-\sigma \cdot ds}{r} = -V_0$, and this is constant wherever V_0 is constant.

Only one distribution of the same quantity of electricity M on the same conductor, removed from the influence of all other electricity, is possible; for, suppose two different values of surface density possible, $\sigma_1 = f_1(x, y, z)$ and $\sigma_2 = f_2(x, y, z)$, then $-\sigma_2 = -f_2(x, y, z)$ is a possible distribution of the charge $-M$. Superpose the distribution $-\sigma_2$ upon the distribution σ_1 so that the total charge shall be equal to zero; then the surface density at every point is $\sigma_1 - \sigma_2$, and this must be zero by what we have just proved, so that $\sigma_1 = \sigma_2$.

Since we may superpose on the same conductor a number of distributions, each one of which is by itself in equilibrium, it is easy to see that if the whole quantity of electricity on any conductor be changed in a given ratio, the density at each point will be changed in the same ratio.

56. Tubes of Force and their Properties. We have seen that a unit of positive electricity concentrated at a point P just outside a conductor would be urged away from the conductor or drawn towards it, according as that point on the conductor which is nearest P is positively or negatively electrified. If we regard lines of force drawn in an electric field as generated by points moving *from* places of higher potential *to* places of lower potential, we may say that a line of force *proceeds from* every point of a conductor where the surface density is positive, and that a line of force *ends at* every point of a conductor where the surface density is negative. No line of force either leaves or enters a conductor at a point where the surface density is zero, and no line of force can start at one point of a conductor where the electrification is positive and return to the same conductor at a point where the electrification is negative. No line of force can proceed from one conductor at a point electrified in any way and enter another conductor at a point where the electrification has the same name as at the starting-point. A line of force never cuts through a conductor so as to come out at the other side, for the force at every point inside a conductor is zero.

Lines and tubes of force are sometimes called in electrostatics lines and tubes of "induction."

When a tube of force joins two conductors, the charges Q_1, Q_2 of the portions S_1, S_2 which it cuts from the two surfaces are

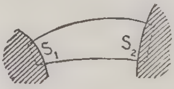

FIG. 40.

made up of equal quantities of opposite kinds of electricity. For if we suppose the tube of force to be arbitrarily prolonged and closed at the ends inside the two conductors, the surface integral of normal force taken over the box thus formed is zero, for the part outside the conductors yields nothing, since the resultant force is tangential to it, and there is no resultant force at any point inside a conductor. It follows, from Gauss's Theorem, that the whole quantity of electricity $(Q_1 + Q_2)$ inside the box must be zero, or $Q_1 = -Q_2$, which proves the theorem. If σ_1 and σ_2 are the average values of the surface densities of the charges on S_1 and S_2 respectively, we have $\sigma_1 S_1 = Q_1$ and $\sigma_2 S_2 = Q_2$, whence

$$\sigma_2 = -\sigma_1 \frac{S_1}{S_2} \qquad [162]$$

The integral taken over any surface, closed or not, of the force normal to that surface is called by some writers the *flow of force* across the surface in question, and by others the *induction* through this surface.

If we apply Gauss's Theorem to a box shut in by a tube of force and the portions S_1, S_2 which it cuts from any two equipotential surfaces, we shall have, if the box contains no electricity,

$$F_2 S_2 - F_1 S_1 = 0, \qquad [163]$$

where F_1 and F_2 are the average values, over S_1 and S_2 respectively, of the normal force taken in the same direction (that in which V decreases) in both cases. In other words, the flow of force across all equipotential sections of a tube of force containing no electricity is the same, or the average force over an equipotential section of an empty tube of force is inversely proportional to the area of the section.

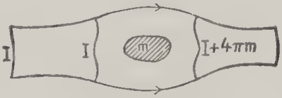

Fig. 41.

When a tube of force encounters a quantity m of electricity (Fig. 41), the flow of force through the tube on passing this electricity is increased by $4\pi m$. If, however, the tube encounters a conductor large enough to close its end completely, a charge m will be found on the conductor just sufficient to reduce to zero the flow of force (I) through the tube. That is,

$$m = -\frac{I}{4\pi}.$$

It is sometimes convenient to consider an electric field to be divided up by a system of tubes of force, so chosen that the flow of force across any equipotential surface of each tube shall be equal to 4π. Such tubes are called *unit tubes ;*[*] for wherever one of them abuts on a conductor, there is always the unit quantity of electricity on that portion of the conductor's surface which the tube intercepts. In some treatises on electricity the term "line of force" is used to represent a unit tube of force, as when a conductor is said to cut a certain number of "lines of force."

It is evident that m unit tubes abut on a surface just outside a conductor charged with m units of either kind of electricity, if the superficial density of the charge has everywhere the same sign. These tubes must be regarded as *beginning* at the conductor if m is positive, and as *ending* there if m is negative. If a conductor is charged at some places with positive electricity and at others with negative electricity, tubes of force will begin where the electrification is positive, and *others* will end where the electrification is negative.

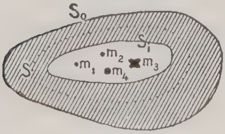

FIG. 42.

It is evident that no tube of force can return into itself.

57. Hollow Conductors. When the nonconducting cavity, shut in by a hollow conductor K (Fig. 42), contains quantities of electricity (m_1, m_2, m_3, etc., or $\sum m$) distributed in any way, but insulated from K, there is induced on the walls of the cavity a charge of electricity algebraically equal in quantity, but opposite in sign, to the algebraic sum of the electricities within the cavity.

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[*] They are sometimes called "unit Faraday tubes," to distinguish them from the more slender tubes of *unit induction*, of which $4\pi m$ start from a body which has a positive charge m.

Call the outside surface of the conductor S_o and its charge M_o, the boundary of the cavity S_i and its charge M_i, and surround the cavity by a closed surface S, every point of which lies within the substance of the conductor, where the resultant force is zero. Now the surface integral of normal force taken over S is zero, so that, according to Gauss's Theorem, the algebraic sum of the quantities of electricity within the cavity and upon S_i is zero. That is,

$$M_i + m_1 + m_2 + m_3 + \cdots = M_i + \sum (m) = 0, \qquad [164]$$

and this is our theorem, which is true whatever the charge on S_o is, and whatever distribution of free electricity there may be outside K. If the distribution of the electricity within the cavity be changed by moving m_1, m_2, etc., to different positions, the *distribution* of M_i on S_i will in general be changed, although its value remains unchanged.

If K has received no electricity from without, its total charge must be zero; that is,

$$M_o = -M_i = \sum (m).$$

If a charge algebraically equal to M be given to K,

$$M_o = M - M_i.$$

The combined effect of $\sum (m)$, the electricity within the cavity, and M_i, the electricity on the walls of the cavity, is at all points without S_i absolutely null. For, if we apply [153] to S, any surface drawn in the conductor so as to enclose S_i, we shall have $D_n V$ everywhere zero, since the potential function is constant within the conductor; this shows that V_1, the potential function due to all the electricity within S, must be zero at all points without S; but S may be drawn as nearly coincident with S_i as we please. Hence our theorem, which shows that, so far as the value of the potential function in the substance of the conductor or outside it, and so far as the arrangement of M_o and of M', any free electricity there may be outside K, are concerned, M_i and $\sum(m)$ might be removed together without changing anything. The potential function at all points outside S_i is to be found by considering only M_o and M'.

If S_i happens to be one of the equipotential surfaces of $\sum(m)$ considered by itself, M_i will be arranged in the same way as a charge of the same magnitude would arrange itself on a conductor whose outside surface was of the shape S_i, if removed from the action of all other free electricity.

The potential function (V_2) due to M_o and M' is constant everywhere within S_o; for if we apply [154_A] to a surface S, drawn within the substance of the conductor as near S_o as we like, we shall have

$$V_s - V_0 = 0,$$

which proves the theorem.

The potential function within the cavity is equal to $V_2 + V_1$, where V_1 is the potential function due to M_i and $\sum(m)$. Of these, V_2 is, as we have seen, constant throughout K and the cavity (Section 31) which it encloses, while V_1 has different values in different parts of the cavity, and is zero within the substance of the conductor.

Suppose now that, by means of an electrical machine, some of the two kinds of electricity existing combined together in a conductor within the cavity be separated, and equal quantities (q) of each kind be set free and distributed in any manner within the cavity.

The value of V_1 within the cavity will probably be different from what it was before, but V_2 will be unchanged; for the quantity of matter in the cavity is unchanged, being now, algebraically considered,

$$\sum(m)+q-q=\sum(m),$$

so that M_i is unchanged, although it may have been differently arranged on S_i, in order to keep the value of V_1 zero within the substance of the conductor. If now a part of the free electricity in the cavity be conveyed to S_i in some way, the substance of the conductor will still remain at the same potential as before. For, if l units of positive electricity and n units of negative electricity be thus transferred to S_i, the whole quantity of free electricity within the cavity will be $\sum(m)-l+n$, and that on S_i will be M_i+l-n: but these are numerically equal, but opposite in sign, and the charge on S_i, if properly arranged, suffices, without drawing on M_o to reduce to zero the value of V_1 in K. Since M_o and M' remain as before, V_2 is unchanged, and the conductor is at the same potential as before. So long as no electricity is introduced into the cavity from *without* K, no electrical changes within the cavity can have any effect outside S_i.

Most experiments in electricity are carried on in rooms, which we can regard as hollows in a large conductor, the earth. V_2, the value of the potential function in the earth and the walls of the room, is not changed by anything that goes on inside the room, where the potential function is $V=V_1+V_2$. Since we are generally concerned, not with the absolute value of the potential function, but only with its variations within the room, and since V_2 remains always constant, it is often convenient to disregard V_2 altogether, and to call V_1 the value of the potential function inside the room. When we do this we must remember that we are taking the value of the potential function in the earth as an arbitrary zero, and that the value of V_1 at a point in the room really measures only the difference between the values of the potential function in the earth and at the point in question. When a conductor A in the room is connected with the walls of the room by a wire, the value of V_1 in A is, of course, zero, and A is said to have been *put to earth.*

58. Induced Charge on a Conductor which is put to Earth. Suppose that there are in a room a number of conductors, viz.: A_1 charged with M_1 units of electricity, and A_2, A_3, A_4, etc., connected with the walls of the room, and therefore at the potential of the earth, which we will take for our zero. If the potential function has the value p_1 inside A_1, every point in the room outside the conductors must have a value of the potential function lying between p_1 and 0, else the potential function must have a maximum or a minimum in empty space. If p_1 is positive, there can be no positive electricity on the other conductors; for if there were, lines of force must start from these conductors and go to places of lower potential; but there are no such places, since these conductors are at potential zero, and all other points of the room at positive potentials. In a similar way we may prove that if p_1 is negative, the electricity induced on the other conductors is wholly positive.

Now let us apply [154_8] to a spherical surface, drawn so as to include A_1 and at least one of the other conductors, but with radius a so small that some parts of the surface shall lie within the room. If we take the point O at the centre of this surface, we shall have

$$4\pi V_2 = \frac{1}{a}\int D_r V \cdot ds + \frac{1}{a^2}\int V ds. \qquad [165]$$

If M is the whole quantity of electricity within the spherical surface, there must be a quantity $-M$ outside the surface, either on the walls of the room or on conductors within the room. The value at O of the potential function, V_2, due to the electricity without the sphere, is less in absolute value than $-\frac{M}{a}$, for it could only be as great as this if all the electricity outside the sphere were brought up to its surface.

By Gauss's Theorem,

$$\int D_r V \cdot ds = -4\pi M,$$

therefore, $$\int V ds = 4\pi a\,[M + aV_2]. \qquad [166]$$

Now, if M_1 is positive, the integral is positive, for all parts of the spherical surface within the room yield positive differentials, and all other parts zero, so that the second side of the equation is positive. But aV_2 is of opposite sign to M, and is less in absolute value; hence, M is positive, and the total amount of negative electricity induced on the other conductors within the spherical surface by the charge on A_1, is numerically less than this charge, unless some one of these conductors surrounds A_1; in which case the induced charge comes wholly on this conductor, while the other conductors, and the walls of the room, are free. Some of the tubes of force which begin at A_1 end on the walls of the room, provided these latter can be reached from A_1 without passing through the substance of any conductor.

59. Coefficients of Induction and Capacity. If a number of insulated conductors, A_2, A_3, A_4, etc., are in a room in the presence of a conductor A_1 charged with M_1 units of electricity, the whole charge on each is zero; but equal amounts of positive and negative electricity are so arranged by induction on each, that the potential function is constant throughout the substance of every one of the conductors.

Let the values of the potential functions in the system of conductors be p_1, p_2, p_3, p_4, etc. Since each conductor except A_1 is electrified, if at all, in some places with positive electricity, and in others with negative electricity, some lines of force must start from, and others end at, every such electrified conductor, so that there must be points in the air about each conductor at lower and at higher potentials than the conductor itself. But the value of the potential function in the walls of the room is zero, and there can be no points of maximum or minimum potential in empty space; so that p_1 must be that value of the potential function in the room most widely different from zero, and p_2, p_3, p_4, etc., must have the same sign as p_1.

The reader may show, if he likes, that both the negative part and the positive part of the zero charge of any conductor, except A_1, is less than M_1.

Let p_{11} be the value of the potential function in a conductor A_1 charged with a single unit of electricity, and standing in the presence of a number of other conductors all uncharged and insulated. Then if p_{12}, p_{13}, p_{14}, etc., are, under these circumstances, the values of the potential functions in the other conductors, A_2, A_3, A_4, etc., the potential functions in these conductors will be $M_1 p_{12}$, $M_1 p_{13}$, $M_1 p_{14}$, etc., if A_1 be charged with M_1 units of electricity instead of with one unit. This is evident, for we may superpose a number of distributions which are singly in equilibrium upon a set of conductors, and get a new distribution in equilibrium where the density is the sum of the densities of the component distributions, and the value of the resulting potential function the sum of the values of their potential functions.

If A_1 be discharged and insulated, and a charge M_2 be given to A_2, the values of the potential functions in the different conductors may be written

$$M_2 p_{21},\ M_2 p_{22},\ M_2 p_{23},\ M_2 p_{24},\ \text{etc.}$$

If now we give to A_1 and A_2 at the same time the charges M_1 and M_2 respectively, and keep the other conductors insulated, the result will be equivalent to superposing the second distribution, which we have just considered, upon the first, and the conductors will be respectively at potentials,

$$M_1 p_{11} + M_2 p_{21},\ M_1 p_{12} + M_2 p_{22},\ M_1 p_{13} + M_2 p_{23},\ \text{etc.} \qquad [167]$$

If all the conductors are simultaneously charged with quantities M_1, M_2, M_3, M_4, etc., of electricity respectively, the value of the potential function on A_k will be

$$V_k = M_1 p_{1k} + M_2 p_{2k} + M_3 p_{3k} + \cdots + M_k p_{kk} + M_n p_{nk}, \qquad [168]$$

Writing this in the form $V_k = a_k + M_k p_{kk}$, we see that if the charges on all the conductors except A_k be unchanged, a_k will be constant, and that every addition of $\frac{1}{p_{kk}}$ units of electricity to the charge of A_k raises the value of the potential function in it by unity. If we solve the n equations like [168] for the charges, we shall get n equations of the form

$$M_k = V_1\,q_{1k} + V_2\,q_{2k} + V_3\,q_{3k} + \cdots + V_k q_{kk} + \cdots + V_n\,q_{nk}, \quad [169]$$

where the q's are functions of the p's.

If all the conductors except A_k are connected with the earth, $M_k = V_k\,q_{kk}$, and q_{kk} is evidently the charge which, under these circumstances, must be given to A_k in order to raise the value of the potential function in it by unity. It is to be noticed that q_{kk} and $\frac{1}{p_{kk}}$ are in general different.

The charge which must be given to a conductor when all the conductors which surround it are in communication with the earth, in order to raise the value of the potential function within that conductor from zero to unity, shall be called the *capacity* of the conductor. It is evident that the capacity of a conductor thus defined depends upon its shape and upon the shape and position of the conductors in its neighborhood.

60. Distribution of Electricity on a Spherical Conductor. Considerations of symmetry show that if a charge M be given to a conducting sphere of radius r, removed from the influence of all electricity except its own, the charge will arrange itself uniformly over the surface, so that the superficial density shall be everywhere $\sigma = \frac{M}{4\pi r^2}$.

The value, at the centre of the sphere, of the potential function due to the charge M on the surface is $\frac{M}{r}$, and, since the potential function is constant inside a charged conductor, this must be the value of the potential function throughout the sphere. If M is equal to r, $\frac{M}{r} = 1$; hence the capacity of a spherical conductor removed from the influence of all electricity except its own, is numerically equal to the radius of its surface.

61. Distribution of a Given Charge on an Ellipsoid. It is evident from the discussion of homœoids in Chapter I. that a charge of electricity arranged (on a conductor) in the form of a shell, bounded by ellipsoidal surfaces similar to each other (and to the surface of the conductor), and similarly placed, would be in equilibrium if the conductor were removed from the action of all electricity except its own. We may use this principle to help us to find the distribution of a given charge on a conducting ellipsoid.

Let us consider a shell of homogeneous matter bounded by two similar, similarly placed, and concentric ellipsoidal surfaces, whose semi-axes shall be respectively a, b, c, and $(1+\alpha)\,a$, $(1+\alpha)\,b$, $(1+\alpha)\,c$. If any line be drawn from the centre of the shell so as to cut both surfaces, the tangent planes to these two surfaces at the points of intersection will be parallel, and the distance between the planes is $p\alpha$, where p is the length of the perpendicular let fall from the centre upon the nearer of the planes.

If ρ is the volume density of the matter of which the shell is composed, the mass of the shell is $M = \frac{4}{3}\pi abc\,[(1+\alpha)^3 - 1]\,\rho$, and the rate at which the matter is spread upon the unit of surface is, at any point, $\sigma = \rho\delta$, where δ is the thickness of the shell measured on the line of force which passes through the point in question. Eliminating ρ from these equations, we have

$$\sigma = \frac{M\delta}{4\pi abc\,[\alpha + \alpha^2 + \frac{1}{3}\alpha^3]}. \qquad [170]$$

If, now, in accordance with the hypothesis that the thickness of the electric charge on a conductor is inappreciable, we make α smaller and smaller, noticing that δ differs from $p\alpha$ by an infinitesimal of an order higher than the first, we shall have for a strictly surface distribution,

$$\sigma = \frac{Mp}{4\pi abc}. \qquad [171]$$

If the equation of the surface of the ellipsoidal conductor is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

we have

$$\frac{1}{p}=\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}},$$

and

$$\frac{c}{p}=\sqrt{c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)+1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}.$$

This last expression shows that, as c is made smaller and smaller, σ approaches more and more nearly the value

$$\frac{M}{4\pi ab\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}, \qquad [172]$$

and this gives some idea of the distribution on a thin elliptical plate whose semi-axes are a and b.

For a circular plate, we may put $a=b$ in the last expression, which gives

$$\frac{M}{4\pi a\sqrt{a^2-r^2}} \qquad [173]$$

for the surface density at a point r units distant from the centre of the plate.

The charge M distributed according to this law on both sides of a circular plate of radius a raises the plate to potential

$$V=\frac{M}{a}\int_0^a\frac{dr}{\sqrt{a^2-r^2}}=\frac{\pi M}{2a},$$

so that the capacity of the plate is

$$\frac{2a}{\pi}. \qquad [174]$$

62. Spherical Condensers. If a conducting sphere A of radius r (Fig. 43) be surrounded by a concentric spherical conducting shell B of radii r_i and r_o and charged with m units of electricity while B is uncharged and insulated, we shall have

(1) the charge m uniformly distributed upon S, the surface of the sphere;

(2) an induced charge $-m$ (Section 57) uniformly distributed upon S_i, the inner surface of B;

(3) a charge $+m$ (since the total charge of B is zero) uniformly distributed on S_o, the outer surface of B.

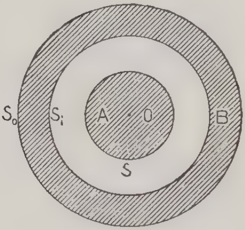

FIG. 43.

The value at the centre of the sphere of the potential function due to all these distributions is $V_A = \frac{m}{r} - \frac{m}{r_i} + \frac{m}{r_o}$, and this is the value of V throughout the conducting sphere. The value of the potential function in B is $V_B = \frac{m}{r_o}$.

If now a charge M be communicated to B, this will add itself to the charge m already existing on S_o, and the charge on S_i will be undisturbed. The values of the potential functions in the conductors are now

$$V_A = \frac{m}{r} - \frac{m}{r_i} + \frac{m+M}{r_o}, \text{ and } V_B = \frac{m+M}{r_o}.$$

If now B be connected with the earth so as to make $V_B = 0$, the charges on S and S_i will be undisturbed, but the charge on S_o will disappear. V_A is now equal to $\frac{m}{r} - \frac{m}{r_i}$.

If A were uncharged, and B had the charge M, this charge would be uniformly distributed upon S_o, for, since the whole charge on S is zero, the whole charge on S_i must be zero also. It is easy to see that S and S_i must both be in a state of nature, for if not, lines of force must start from S and end at S_i, and others start at S_i and end at S, which is absurd.

If A were put to earth by means of a fine insulated wire passing through a tiny hole in B, and if B were insulated and charged with M units of electricity, we should have a charge x on S, a charge $-x$ on S_i, and a charge $M+x$ on S_o. To find x, we need only remember that $V_A=\frac{x}{r}-\frac{x}{r_i}+\frac{x}{r_o}+\frac{M}{r_o}=0$, whence x may be obtained.

If B be put to earth, and A be connected by means of the fine wire just mentioned, with an electrical machine which keeps its prime conductor constantly at potential V_1, A will receive a charge y and will be put at potential V_1. To find y, it is to be noticed that there is a charge $-y$ on S_i, and no charge on S_o, which is put to earth. $V_A=\frac{y}{r}-\frac{y}{r_i}=V_1$, whence y may be obtained.

If $r=99$ millimeters and $r_i=100$ millimeters, $y=9900\ V_1$.

If a sphere, equal in size to A but having no shell about it, were connected with the same prime conductor, it too would receive a charge z sufficient to raise it to potential V_1, and z would be determined by the equation $V_1=\frac{z}{r}$. If $r=99$, we have $z=99\ V_1$; hence we see that A, when surrounded by B at potential zero, is able to take one hundred times as great a charge from a given machine as it could take if B were removed. In other words, B increases A's capacity one hundred fold. A and B together constitute what is called a *condenser*.

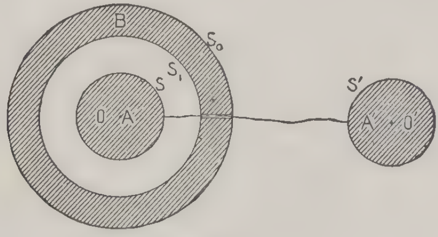

FIG. 44.

If A of the condenser AB, both parts of which are supposed uncharged, be connected by a fine wire (Fig. 44) with a sphere

A' which has the same radius as A, and is charged to potential V_1, A and A' will now be at the same potential [V_2], and A will have the charge x, and A' the charge y. The total quantity of electricity on A' at first was rV_1, so that $x + y = rV_1$, and

$$V_2 = \frac{y}{r} = \frac{x}{r} - \frac{x}{r_i} + \frac{x}{r_o},$$

whence x and y may be found.

The reader may study for himself the electrical condition of the different parts of two equal spherical condensers (Fig. 45,,

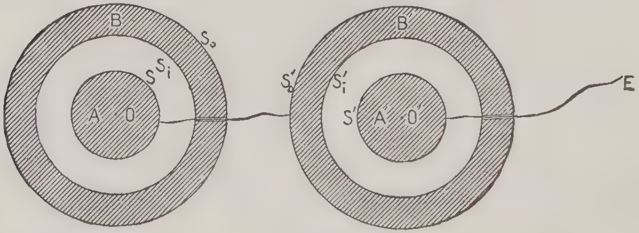

FIG. 45.

of which the outer surface S_o of one is connected with an electric machine at potential V_1, and the inside of the other, S', is connected with the earth. The two condensers, which are supposed to be so far apart as to be removed from each other's influence, illustrate the case of two Leyden jars arranged in cascade.

63. Condensers made of Two Parallel Conducting Plates. Suppose two infinite conducting planes A and B to be parallel to each other at a distance a apart; choose a point of the plane A for origin, and take the axis of x perpendicular to the planes, so that their equations shall be $x = 0$ and $x = a$. Let the planes be charged and kept at potentials V_A and V_B respectively. It is evident from considerations of symmetry that the potential function at the point P between the two planes depends only upon P's x coördinate, so that

$$D_y V = 0, \quad D_z V = 0, \quad D_y^2 V = 0, \quad D_z^2 V = 0.$$

Laplace's Equation gives, then,

$$D_x^2 V = 0,$$

whence $$D_x V = C, \text{ and } V = Cx + D.$$

If $x = 0$, $V = V_A$; and if $x = a$, $V = V_B$; so that

$$V = (V_B - V_A)\frac{x}{a} + V_A, \text{ and } D_x V = \frac{V_B - V_A}{a}.$$

The lines of force are parallel between the planes, and the surface densities of the charges on A and B are

$$\frac{V_A - V_B}{4\pi a} \text{ and } \frac{V_B - V_A}{4\pi a} \text{ respectively.}$$

If we take a portion of area S out of the middle of each plate, there will be a quantity of electricity on S_A equal to $\frac{S(V_A - V_B)}{4\pi a}$, and an equal quantity of the other kind of electricity on S_B. The force of attraction between S_A and S_B will be $2\pi\sigma^2 \cdot S$, or

$$\frac{S}{8\pi}\,\frac{(V_B - V_A)^2}{a^2}.$$

If S_B be put to earth, the charge that must be given to S_A in order to raise it to potential unity is

$$\frac{S}{4\pi a}.$$

In other words, the capacity of S_A is inversely proportional to the distance between the plates.

In the case of two thin conducting plates placed parallel to and opposite each other, at a distance small compared with their areas, the lines of force are practically parallel except in the immediate vicinity of the edges of the plates ;[*] and we may infer

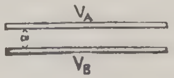

FIG. 46.

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[*] See Maxwell's *Treatise on Electricity and Magnetism,* Vol. I. Fig. XII.

from the results of this section that the capacity of a condenser consisting of two parallel conducting plates of area S, separated by a layer of air of thickness a, when one of its plates is put to earth is very approximately $\frac{S}{4\pi a}$ for large values of $\frac{S}{a}$.

64. Capacity of a Long Cylinder surrounded by a Concentric Cylindrical Shell. In the case of an infinite, conducting cylinder of radius r_i, kept at potential V_i and surrounded by a concentric conducting cylindrical shell of radii r_o and r', kept at potential V_o, we have symmetry about the axis of the cylinder, so that $D_\phi V = 0$, and Laplace's Equation reduces to the form

$$D_r^2 V + \frac{D_r V}{r} = 0,$$

whence, for all points of empty space between the cylinder and its shell,

$$V = C + D \log r.$$

But $V = V_i$ when $r = r_i$, and $V = V_o$ when $r = r_o$,

hence
$$V = \frac{V_i \log \frac{r_o}{r} + V_o \log \frac{r}{r_i}}{\log \frac{r_o}{r_i}}, \qquad [175]$$

and
$$D_r V = \frac{(V_o - V_i)}{\log \frac{r_o}{r_i}} \cdot \frac{1}{r}.$$

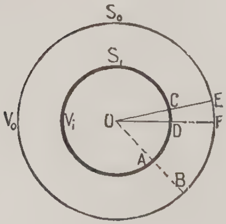

FIG. 47.

The surface densities of the electricity on the outer surface of the cylinder and the inner surface of the shell are respectively

$$\frac{V_i - V_o}{4\,\pi r_i \log \dfrac{r_o}{r_i}} \text{ and } \frac{V_o - V_i}{4\,\pi r_o \log \dfrac{r_o}{r_i}},$$

so that the charge on the unit of length of the cylinder is $\dfrac{V_i - V_o}{2\log \dfrac{r_o}{r_i}}$, and the charge on the corresponding portion of the inner surface of the shell is the negative of this. We may find the capacity of the unit length of the cylinder by putting $V_o = 0$ and $V_i = 1$, whence capacity $= \dfrac{1}{2\log \dfrac{r_o}{r_i}}$.

If r_o in this expression is made very large, the capacity of the cylinder will be very small.

In the case of a fine wire connecting two conductors, r_i will be very small, and there will be no conducting shell nearer than the walls of the room, so that the capacity of such a wire is plainly negligible.

65. Charge induced on a Sphere by a Charge at an Outside Point. The value at any point P of the potential function due to m_1 units of positive electricity concentrated at a point A_1, and m_2 units of negative electricity concentrated at a point A_2, is

$$V = \frac{m_1}{r_1} - \frac{m_2}{r_2}, \text{ where } r_1 = A_1P \text{ and } r_2 = A_2P.$$

It is easy to see that if m_1 is greater than m_2, so that

$$m_1 = \lambda m_2$$

where $\lambda > 1$, V will be equal to zero all over a certain sphere which surrounds A_2.

If (Fig. 48) we let $A_1A_2 = a$, $A_1O = \delta_1$, $A_2O = \delta_2$, $OD = r$, it is easy to see that

$$\delta_1 = \frac{\lambda^2 a}{\lambda^2 - 1}, \quad \delta_2 = \frac{a}{\lambda^2 - 1}, \quad r^2 = \frac{a^2\lambda^2}{(\lambda^2 - 1)^2} = \delta_1\delta_2, \quad \lambda^2 = \frac{\delta_1}{\delta_2},$$

and
$$a = \frac{\delta_1^2 - r^2}{\delta_1} = \frac{r^2 - \delta_2^2}{\delta_2}. \qquad [176]$$

If PR represents the force f_1 due to the electricity at A_1, and PQ the force f_2 due to the electricity at A_2, the line of action of the resultant force F (represented by PL) must pass through the centre of the sphere, since the surface of the sphere is equipotential.

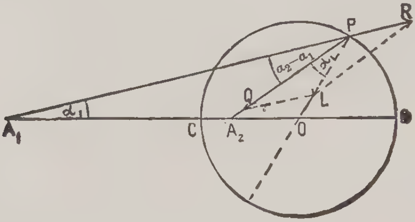

FIG. 48.

The triangles A_1PO and A_2PO are mutually equiangular, for they have a common angle A_1OP, and the sides including that angle are proportional ($r^2 = \delta_1\delta_2$). Hence, from the triangles QPL and A_1PA_2, by the Theorem of Sines,

$$\frac{f_1}{\sin\alpha_1} = \frac{f_2}{\sin\alpha_2} = \frac{F}{\sin(\alpha_2 - \alpha_1)}, \qquad [177]$$

$$\frac{r_1}{\sin\alpha_2} = \frac{r_2}{\sin\alpha_1} = \frac{a}{\sin(\alpha_2 - \alpha_1)}, \qquad [178]$$

whence
$$F = \frac{af_1}{r_2} = \frac{am_1}{r_2 r_1^2} = \frac{a\lambda m_1}{r_1^3}. \qquad [179]$$

Now, according to Section 49, we may distribute upon the spherical surface just considered a quantity m_2 of negative electricity in such a way that the effect of this distribution at all points outside the sphere shall be equal to the effect of the charge $-m_2$ concentrated at A_2, and the effect at points within the sphere shall be equal and opposite to the effect of the charge m_1 concentrated at A_1. Since F is the force at P in the direction of the interior normal to the sphere, we shall accomplish this if we make the surface density at every point equal to σ, where

$$4\pi\sigma = -F = \frac{-a\lambda m_1}{r_1^3} = \frac{-(\delta_1^2 - r^2)\,m_1}{r r_1^3}; \qquad [180]$$

and if we now take away the charge at A_2, the value of the potential function throughout the space enclosed by our spherical surface, and upon the surface itself, will be zero. If the spherical surface were made conducting, and were connected with the earth by a fine wire, there would be no change in the charge of the sphere, and we have discovered the amount and the distribution of the electricity induced upon a sphere of radius r, connected with the earth by a fine wire and exposed to the action of a charge of m_1 units of positive electricity concentrated at a point at a distance δ_1 from the centre of the sphere.

If now we break the connection with the earth, and distribute a charge m uniformly over the sphere in addition to the present distribution, the potential function will be constant (although no longer zero) within the sphere, and we have a case of equilibrium, for we have superposed one case of equilibrium (where there is a uniform charge on the sphere and none at A_1) upon another. The whole charge on the sphere is now

$$M = m - m_2 = m - \frac{m_1 r}{\delta_1},$$

and the value of the potential function within it and upon the surface,

$$V = \frac{M}{r} + \frac{m_1}{\delta_1} = \frac{m}{r}.$$

If the conducting sphere were at the beginning insulated and uncharged, we should have $M = 0$, and therefore

$$\sigma = \frac{m_1}{4\pi r}\left(\frac{1}{\delta_1} - \frac{\delta_1^2 - r^2}{r_1^3}\right), \text{ and } V = \frac{m_1}{\delta_1}. \qquad [181]$$

If we have given that the conducting sphere, under the influence of the electricity concentrated at A_1 is at potential V_1, we know that its total charge must be $V_1 r - \frac{m_1 r}{\delta_1}$, and its surface density

$$= \frac{1}{4\pi r}\left(V_1 - \frac{(\delta_1^{\,2} - r^2)\, m_1}{r_1^{\,3}}\right). \qquad [182]$$

It is easy to see that the sphere and its charge will be attracted toward A_1 with the force

$$\frac{m_1 r}{\delta_1}\left(\frac{m_1 \delta_1^{\,2}}{(\delta_1^{\,2} - r^2)^2} - \frac{V_1}{\delta_1}\right); \qquad [183]$$

and the student should notice that, under certain circumstances, this expression will be *negative* and the force repulsive.

If $m_1 = m_2$, the surface of zero potential is an infinite plane, and our equations give us the charge induced on a conducting plane by a charge at a point outside the plane.

The method of this section enables us to find also the capacity of a condenser composed of two conducting cylindrical surfaces, parallel to each other, but eccentric; for a whole set of the equipotential surfaces due to two parallel, infinite straight lines, charged uniformly with equal quantities per unit of length of opposite kinds of electricity, are eccentric cylindrical surfaces surrounding one of the lines, A_2, and leaving the other line, A_1, outside. We may therefore choose two of these surfaces, distribute the charge of A_1 on the outer of these, and the charge of A_2 on the inner, by the aid of the principles laid down in Section 49, so as to leave the values of the potential function on these surfaces the same as before. These distributions thus found will remain unchanged if the equipotential surfaces are made conducting.

The reader who wishes to study this method more at length should consult, under the head of Electric Images, the treatises of Cumming, Maxwell, Mascart, Tarleton, and Watson and Burbury, as well as original papers on the subject by Murphy in the *Philosophical Magazine*, 1833, p. 350, and by Sir W. Thomson in the *Cambridge and Dublin Mathematical Journal* for 1848.

66. The Energy of Charged Conductors. If a conductor of capacity C, removed from the action of all electricity except its own, be charged with M_1 units of electricity, so that it is at potential $V_1 = \frac{M_1}{C}$, the amount of work required to bring up to the conductor, little by little, from the walls of the room, the additional charge ΔM, is ΔW, which is greater than $V_1 \cdot \Delta M$ or $\frac{M_1}{C} \cdot \Delta M$, and less than $(V_1 + \Delta_M V) \cdot \Delta M$ or $\frac{M_1 + \Delta M}{C} \cdot \Delta M$.

If the charge be increased from M_1 to M_2 by a constant flow, the amount of work required is evidently

$$\int_{M_1}^{M_2} \frac{M\,dM}{C} = \frac{M_2^2 - M_1^2}{2\,C} = \frac{C}{2}(V_2^2 - V_1^2). \qquad [184]$$

The work required to bring up the charge M to the conductor at first uncharged is then

$$\frac{M^2}{2\,C} = \frac{C V^2}{2} = \frac{MV}{2}. \qquad [185]$$

This is evidently equal to the potential energy of the charged conductor, and this is independent of the method by which the conductor has been charged.

If, now, we have a series of conductors A_1, A_2, A_3, etc., in the presence of each other at potentials V_1, V_2, V_3, etc., and having respectively the charges M_1, M_2, M_3, etc., and if we change all the charges in the ratio of x to 1, we shall have a new state of equilibrium in which the charges are $x M_1$, $x M_2$, $x M_3$, etc.; and the values of the potential functions within the conductors are $x V_1$, $x V_2$, $x V_3$, etc. The work (ΔW) required to increase the charges in the ratio $x + \Delta x$ instead of in the ratio x is greater than

$$(M_1 \Delta x)(x V_1) + (M_2 \Delta x)(x V_2) + (M_2 \Delta x)(x V_2) + \text{etc.},$$

or

$$x \Delta x [M_1 V_1 + M_2 V_2 + M_3 V_3 + \text{etc.}],$$

and less than

$$(x + \Delta x) \Delta x [M_1 V_1 + M_2 V_2 + M_3 V_3 + \text{etc.}];$$

hence the whole amount of work required to change the ratio from $\frac{x_1}{1}$ to $\frac{x_2}{1}$ is

$$W_2 - W_1 = \frac{x_2^2 - x_1^2}{2}[M_1V_1 + M_2V_2 + M_3V_3 + \text{etc.}]. \quad [186]$$

If in this equation we put $x_1 = 0$ and $x_2 = 1$, we get for the work required to charge the conductors from the neutral state to potentials V_1, V_2, V_3,

$$W = \tfrac{1}{2}[M_1V_1 + M_2V_2 + M_3V_3 + \cdots] = \tfrac{1}{2}\sum(MV), \quad [187]$$

a particular case of the general formula stated in Section 27.

The work required to make any combination of changes of charge on any system of fixed conductors is evidently equal to the difference between the intrinsic energies of the system in its original and final states. If V_k, V_k' represent the initial and final potentials on the kth conductor, and e_k and e_k' the original and final charges,

$$E' - E = \tfrac{1}{2}\sum V_k'e_k' - \tfrac{1}{2}\sum V_ke_k.$$

Since the final energy is independent of the manner in which the changes are produced, we may suppose that the changes take place gradually and at the same relative rate for all the conductors, so that at any instant the charge of each conductor has received the same fraction of its whole increment or decrement that every other conductor has received, it being understood that in the general case some charges will be increased and others decreased. At the instant when the change accomplished is to the whole change as $x : 1$, the charge of the kth conductor is $e_k + x(e_k' - e_k)$, and the value of the potential function in this conductor is $V_k + x(V_k' - V_k)$. In order to increase x by Δx, the charge must be increased by the amount $\Delta x(e_k' - e_k)$, and to bring this up from infinity an amount of work equal approximately to

$$(e_k' - e_k)[V_k + x(V_k' - V_k)]\Delta x$$

must be done, and for the whole system the corresponding work is $\sum (e_k' - e_k)[V_k + x(V_k' - V_k)]\Delta x$. To find the work required to bring about the whole change, this expression must be integrated with respect to x between the limits 0 and 1. This process yields

$$E' - E = \tfrac{1}{2}\sum (V_k + V_k')(e_k' - e_k),$$

and by comparing this with the result stated above we learn that we may also write

$$E' - E = \tfrac{1}{2}\sum (V_k' - V_k)(e_k' + e_k).$$

We learn incidentally that $\sum e_k' V_k = \sum e_k V_k'$, and we see that if all but two (A_1, A_2) of any system of conductors are either put to earth or are insulated and without charge,

$$e_1' V_1 + e_2' V_2 = e_1 V_1' + e_2 V_2'.$$

If $e_1 = 1$, $e_2 = 0$, $e_1' = 0$, $e_2' = 1$, $V_2 = V_1'$, and if $V_1 = 1$, $V_2 = 0$, $V_1' = 0$, $V_2' = 1$, $e_1' = e_2$, so that a unit charge given to A_1, while A_2 is uncharged and insulated, raises A_2 to the same potential that A_1 would have if it were uncharged and insulated while A_2 had a unit charge; and the same quantity of electricity is induced on A_2 when it is put to earth, while A_1 is charged to potential unity as would be induced on A_1 if it were put to earth and A_2 charged to potential unity. Using the notation of Section 59, this shows that $p_{rk} = p_{kr}$ and that $q_{rk} = q_{kr}$; we may write, therefore,

$$\begin{aligned} E &= \tfrac{1}{2}\sum_k e_k \sum_r e_r p_{rk} = \tfrac{1}{2}\sum_k V_k \sum_r V_r q_{rk} \\ &= \tfrac{1}{2}(p_{11}e_1^2 + p_{22}e_2^2 + p_{33}e_3^2 + \cdots + p_{nn}e_n^2) \\ &\qquad + p_{12}e_1e_2 + p_{13}e_1e_3 + \cdots + p_{23}e_2e_3 + p_{34}e_3e_4 + \cdots \\ &= \tfrac{1}{2}(q_{11}V_1^2 + q_{22}V_2^2 + q_{33}V_3^2 + \cdots + q_{nn}V_n^2) \\ &\qquad + q_{12}V_1V_2 + q_{13}V_1V_3 + \cdots + q_{23}V_2V_3 \\ &\qquad + q_{34}V_3V_4 + \cdots. \end{aligned}$$

If the conductors are fixed so that the p's and q's are constant, we may learn from differentiating this last equation that, if all the charges but e_k are kept constant, $D_{e_k}E = V_k$, and if the values of the potential function in all but one of the conductors (the kth) are unchanged $D_{V_k}E = e_k$.

If the system changes its configuration, the p's and q's are in general changed, and we learn that if the charges are kept constant during the change,

$$\Delta' E = \tfrac{1}{2}\sum_k e_k \sum_r e_r \Delta p_{rk};$$

but that if by suitable changes in the charges the potentials are unchanged,

$$\Delta'' E = \tfrac{1}{2}\sum_k V_k \sum_r V_r \Delta q_{rk}.$$

In the latter case, ΔV_k, or $\sum \Delta(e_r p_{rk}) = 0$, so that $\sum e_k \Delta V_k$, or

$$\sum\sum(e_k e_r \Delta p_{rk} + e_k p_{rk}\Delta e_r + e_r \Delta p_{rk}\Delta e_k),$$

or $$2\,\Delta' E + 2\,\Delta'' E + \sum\sum e_r \Delta p_{rk}\Delta e_k = 0.$$

If, therefore, ϕ is any coördinate, which defines the configuration,

$$\underset{\Delta\phi \doteq 0}{\text{limit}}\left(\frac{\Delta' E}{\Delta\phi}\right) = -\underset{\Delta\phi \doteq 0}{\text{limit}}\left(\frac{\Delta'' E}{\Delta\phi}\right), \text{ or } D_\phi' E + D_\phi'' E = 0.$$

A system of conductors with constant charges when left to itself tends to obey the urgings of the reciprocal forces between its parts, and therefore to diminish its intrinsic energy. If, in this case, the single coördinate ϕ is free to change and is increased by $\Delta\phi$, the energy after the change is $E + \Delta' E$, where $\Delta' E$ is really negative. The mechanical work done by the forces is $-D_\phi E \cdot \Delta\phi$. If, now, ϕ had been changed as before by the same small increment, $\Delta\phi$, while the potentials were kept constant by bringing up to each conductor from without the necessary quantity of electricity, the energy after the change would have been $E + \Delta'' E$, where

$\Delta''E$ is really positive. The energy has therefore increased by an amount practically equal to the former loss. Practically the same amount of mechanical work has been done as before, and enough energy has been introduced from without to do this work and to add an equivalent amount besides this to the potential energy of the system. The contribution, therefore, from outside sources is about $2\,\Delta''E$. These statements applied to a *small* change in ϕ are based on the exact equation $D_\phi'E = -D_\phi''E$, proved above.

67. If a series of conductors A_1, A_2, A_3, etc., are far enough apart not to be exposed to inductive action from one another, and have capacities C_1, C_2, C_3, etc., and charges M_1, M_2, M_3, etc., so as to be at potentials V_1, V_2, V_3, etc., where $M_1 = C_1 V_1$, $M_2 = C_2 V_2$, $M_3 = C_3 V_3$, etc., we may connect them together by means of fine wires whose capacities we may neglect, and thus obtain a single conductor of capacity

$$C_1 + C_2 + C_3 + \cdots = \sum(C).$$

The charge on this composite conductor is evidently

$$M_1 + M_2 + M_3 + \cdots = \sum(M);$$

and if we call the value of the potential function within it V, we shall have

$$V \cdot \sum(C) = \sum(M);$$

whence

$$V = \frac{C_1 V_1 + C_2 V_2 + C_3 V_3 + \cdots}{C_1 + C_2 + C_3 + \cdots}, \qquad [188]$$

a formula obtained, it is to be noticed, on the assumption that the conductors do not influence each other.

The energy of the separate charged conductors before being connected together was

$$W = \tfrac{1}{2}(M_1 V_1 + M_2 V_2 + M_3 V_3 + \cdots) = \tfrac{1}{2}\left(\frac{M_1^2}{C_1} + \frac{M_2^2}{C_2} + \frac{M_3^2}{C_3} + \cdots\right)$$

$$= \tfrac{1}{2}\sum\left(\frac{M^2}{C}\right), \qquad [189]$$

and the energy of the composite conductor is

$$W' = \frac{\tfrac{1}{2}(M_1 + M_2 + M_3 + \cdots)(C_1 V_1 + C_2 V_2 + C_3 V_3 + \cdots)}{C_1 + C_2 + C_3 + \cdots}$$

$$= \frac{\tfrac{1}{2}(M_1 + M_2 + M_3 + \cdots)^2}{C_1 + C_2 + C_3 + \cdots} = \frac{\tfrac{1}{2}\left[\sum(M)\right]^2}{\sum(C)}, \qquad [190]$$

which is always less than E, unless the separate conductors were all at the same potential in the beginning.

68. Specific Inductive Capacity. In all our work up to this time we have supposed conductors to be separated from each other by electrically indifferent media, which simply prevent the passage of electricity from one conductor to another. We have no reason to believe, however, that such media exist in nature. Experiment shows, for instance, that the capacity of a given spherical condenser depends essentially upon the kind of insulating material used to separate the sphere from its shell, so that this material, without conducting electricity, modifies the action of the charges on the conductors. Insulators, when considered as transmitting electric action, are sometimes called *dielectrics.*

Given two condensers of any shape, geometrically alike in all respects, with plates separated in the one case by a homogeneous dielectric, A, and in the other case by another homogeneous dielectric, B, the ratio of the capacities is found to be the same whatever the shape or dimensions of the condensers when these same two dielectrics are used. If this ratio is unity, the dielectrics are said to have the same *electrical inductivity* or the same *specific inductive capacity.* If the ratio of the capacities of the first and second condensers is n, A is said to have an inductivity n times as great as that of B. The electrical inductivity of dry air at the standard pressure and temperature being chosen as a standard, the electrical inductivities of all other known substances are positive quantities which in the case of any one specimen, though somewhat dependent upon conditions of temperature and pressure, may be considered independent of the electrical stress to which the substance may be exposed. The letter μ is often used to represent the inductivity of a medium. It is generally assumed, for the sake of definiteness, that outside all the material media upon which we can experiment, the ether extends indefinitely in all directions and the inductivity of the ether is assumed to be sensibly the same as that of air under standard conditions. We cannot expect that a non-homogeneous dielectric will have the same inductivity throughout, so that in the general case we must assume that μ is a function of the space coördinates. The vector formed by multiplying the force by the scalar quantity $\frac{\mu}{4\pi}$ is sometimes called the *displacement.* The force is occasionally called the *electrical intensity,* or the *electromotive intensity.*

We may best sum up the results of experiments upon the behavior of dielectrics in electric fields by stating some general equations which may be used in solving any problem. We shall find it convenient to write down first, for the sake of comparison, the simplified forms of these equations which we have shown to be characteristic of the electric field about any distribution of electricity when air is the only dielectric.

If X, Y, Z are the force components parallel to the axes, and if V is the potential function, so that

$$X = -D_xV, \quad Y = -D_yV, \quad Z = -D_zV,$$

we know that when $\mu = 1$,

$$(1) \quad D_xX + D_yY + D_zZ = +4\pi\rho,$$

except at surfaces where ρ is discontinuous.

(2) The surface integral of the normal (outward) component of the force taken over any closed surface is equal to 4π times the amount of matter (algebraically reckoned) within the surface; or $\int N dS = 4\pi M.$

(3) At a charged surface all the tangential components of the force are continuous, but the normal components are discontinuous in the manner indicated by the equation

$$N_1 + N_2 = + 4\pi\sigma,$$

where N_1 and N_2 represent the normal force components taken away from the surface on both sides. If the charged surface is not equipotential, the lines of force which cross it are in general refracted; for, if ϕ_1 is the angle which a line of force in reaching the surface makes with its normal, ϕ_2 the angle which the same line makes with the normal on leaving the surface on the other side, and, if T_1 and T_2 are the tangential components of the force, $T_1 = - N_1 \tan\phi_1$, $T_2 = N_2 \tan\phi_2$, and since $T_1 = T_2$, $N_2 \tan\phi_2 + N_1 \tan\phi_1 = 0$, or, since the normal component is discontinuous,

$$(4\pi\sigma - N_1)\tan\phi_2 + N_1 \tan\phi_1 = 0.$$

(4) V so vanishes at infinity that rV and $r^2 D_r V$ have finite limits.

If we now introduce a new vector (called the *induction*) equal to the product of the scalar point function μ and the force, we may write down a set of equations, very like those which we have just enumerated and equivalent to them when $\mu = 1$, which will give the force components and the potential function in terms of the charges when μ is different from unity and (in the general case) determined by different analytic functions of the space coördinates in different portions of space.

In general,

$$(1) \qquad D_x(\mu X) + D_y(\mu Y) + D_z(\mu Z) = + 4\pi\rho \qquad [191]$$

at every point in space, except at surfaces where either ρ or μ is discontinuous. Since in all cases $X = - D_x V$, $Y = - D_y V$, $Z = - D_z V$, this equation may be written

$$D_x(\mu D_x V) + D_y(\mu D_y V) + D_z(\mu D_z V) = - 4\pi\rho. \qquad [192]$$

In a dielectric of uniform inductivity it becomes

$$\mu \nabla^2 V = -4 \pi \rho.$$

(2) The integral, taken over any closed surface, of the outward normal component of the induction is equal to 4π times the amount of matter within the surface, or

$$\int \mu N dS = 4 \pi M. \qquad [193]$$

(3) If the surface of separation between two different dielectrics which are in contact with each other has a charge of superficial density, σ, all the force components tangent to the surface are continuous. If μ_1 and μ_2 are the inductivities of the two media, the normal component of the induction is discontinuous in the manner indicated by the equation

$$\mu_1 N_1 + \mu_2 N_2 = +4 \pi \sigma,$$

or

$$\mu_1 D_{n_1} V + \mu_2 D_{n_2} V = -4 \pi \sigma. \qquad [194]$$

If this surface has no charge, $\sigma = 0$, and the normal component of the induction is continuous, though the normal force component is discontinuous: evidently, the law of refraction of the lines of force is, in this case, $\tan \phi_1 : \mu_1 = \tan \phi_2 : \mu_2$. Whether or not σ is zero, $N_1 \tan \phi_1 + N_2 \tan \phi_2 = 0$. At a charged surface where the dielectric is continuous,

$$\mu (N_1 + N_2) = 4 \pi \sigma.$$

(4) V is everywhere continuous, and it so vanishes at infinity that rV and $r^2 D_r V$ have finite limits. The first derivatives of V are everywhere continuous, except at charged surfaces and surfaces where the inductivity is discontinuous: here the tangential derivatives of V are continuous and the normal derivatives have the properties just discussed. The lines of force and the lines of induction are coincident. It is well to notice that what we have here called the induction and what is usually called induction in perfectly hard magnets are different special cases, as will be shown later on, of a

much more complex vector which appears in some general problems.

It is easy to prove with the help of [149] a series of theorems concerning the potential function analogous to those already found for the case where $\mu = 1$. For instance: if the closed surface S_1 shuts in the closed surface S_2, there cannot be two different functions, V and V', which (1) between S_1 and S_2 satisfy the equation

$$D_x(\mu D_x w) + D_y(\mu D_y w) + D_z(\mu D_z w) = 0,$$

where μ is a given, everywhere positive, analytic function of the space coördinates, (2) are continuous in that region with their first derivatives, and (3) are equal at every point of S_1 and S_2. Assuming, for the sake of argument, that two such functions exist, we may call their difference u and note that u and its first derivatives are continuous between S_1 and S_2, and that u vanishes at every point of these surfaces. Since u satisfies the equation

$$D_x(\mu D_x u) + D_y(\mu D_y u) + D_z(\mu D_z u) = 0$$

between S_1 and S_2, we may conveniently make $\lambda = \mu$, $U = V = u$ in [149], for both integrals in the second member of the equation vanish, and we learn that

$$\iiint \mu[(D_x u)^2 + (D_y u)^2 + (D_z u)^2]\,dx\,dy\,dz = 0$$

when extended over the region in question. Since μ is positive, and the integrand can never be negative,

$$D_x u = D_y u = D_z u = 0,$$

and u is a constant. But $u = 0$ on S_1 and S_2, hence V and V' are identical.

If, while satisfying conditions (1) and (2), V and V' are required to have equal normal derivatives at every point of S_1 and S_2, it is easy to prove in a similar manner that one can differ from the other only by a constant.

With given values of the volume density in given regions of space, and with given values of the superficial density on given surfaces, the force components and the potential function are, in general, different when $\mu = 1$ and when μ is different from 1, and, if the dielectric is heterogeneous with surfaces of discontinuity in μ, not equipotential surfaces, the forms of the lines of force are very different in the two cases.

If the dielectric of a given condenser, the plates of which are the surfaces S_1 and S_2, is air, and if these plates have given charges, V must satisfy Laplace's Equation between S_1 and S_2, while at every part of the condenser plate $D_n V = -4\pi\sigma$. If, now, a homogeneous dielectric of inductivity μ be substituted for the air, the new potential function V' satisfies Laplace's Equation between S_1 and S_2 (since μ is constant and ρ is zero), and at every point of S_1 or S_2, $D_n V' = \dfrac{4\pi\sigma}{\mu}\cdot$ Now V/μ satisfies all these last conditions, and since two functions which do so can at most differ by a constant, we may write

$$V' = V/\mu + C.$$

The force in any direction at any point in the dielectric is $1/\mu$ as great in the second case as in the first. If S_1 and S_2, instead of having given charges, had been kept at the given potentials V_1 and V_2, the density of the charge at any point of either plate would have been μ times as great in the second case as in the first, while the potential function (and the force) would have had the same value at every point, whichever dielectric was used. The capacity of the condenser is, in this case, equal to

$$\mu \int D_n V dS_1 / 4\pi (V_2 - V_1).$$

The generalized form [192] of Poisson's Equation, when expressed in terms of the orthogonal curvilinear coördinates u, v, w as independent variables, becomes

$$\mu[h_u^2 \cdot D_u^2 V + h_v^2 \cdot D_v^2 V + h_w^2 \cdot D_w^2 V + D_u V \cdot \nabla^2 u + D_v V \cdot \nabla^2 v + D_w V \cdot \nabla^2 w]$$
$$+ (D_u V \cdot D_x u + D_v V \cdot D_x v + D_w V \cdot D_x w)(D_u \mu \cdot D_x u + D_v \mu \cdot D_x v + D_w \mu \cdot D_x w)$$
$$+ (D_u V \cdot D_y u + D_v V \cdot D_y v + D_w V \cdot D_y w)(D_u \mu \cdot D_y u + D_v \mu \cdot D_y v + D_w \mu \cdot D_y w)$$
$$+ (D_u V \cdot D_z u + D_v V \cdot D_z v + D_w V \cdot D_z w)(D_u \mu \cdot D_z u + D_v \mu \cdot D_z v + D_w \mu \cdot D_z w) = -4\pi\rho.$$

If μ is a function of one of the coördinates, u, only, the family of surfaces on which u is constant are possible equipotential surfaces due to a distribution of electricity in this dielectric, provided the special form of the equation just stated, obtained by putting

$$D_v \mu = D_w \mu = D_v V = D_w V = \rho = 0,$$

that is, provided the equation

$$D_u^2 V + \left(\frac{\nabla^2 u}{h_u^2} + \frac{D_u \mu}{\mu}\right) D_u V = 0,$$

involves only u. Now $D_u \mu / \mu$ is, by hypothesis, a function of u only, so that the condition is that the ratio of $\nabla^2 u$ to h_u^2 shall be independent of v and w, and this is the condition (Section 35) that must be satisfied when the dielectric is air, in order that the surfaces upon which u is constant may be possible equipotential surfaces.

It is easy to see that if the space between two equipotential surfaces in air about a distribution of electricity be filled with a dielectric the inductivity of which is either constant or else a function only of the parameter of the original equipotential surfaces, the new equipotential surfaces will coincide with the old ones, though the value of the potential function on any particular surface will generally be changed.

If in [149] we make $U = V$, the potential function due to any distribution of electricity, and if we make $\lambda = \mu$, we may apply the equation to all space after we have enclosed by pairs of new surfaces all surfaces of discontinuity of μ, ρ, or $D_n V$, and learn that the intrinsic energy of the distribution is equal to

$$\frac{1}{8\pi}\int\int\int \mu[(D_x V)^2 + (D_y V)^2 + (D_z V)^2]\,dx\,dy\,dz$$

extended over all space.

When the potential function, V, due to a given distribution (ρ, σ) of electricity with any given set of dielectrics has been found, we may ask what distribution (ρ', σ') of electricity would have given this same potential function if all the dielectrics had been displaced by homogeneous air. The distribution (ρ', σ') is called the *apparent charge* to distinguish it from the distribution (ρ, σ) which is sometimes called the *real* or the *intrinsic charge.* From the apparent charge when found, V might be calculated by means of the familiar integrals

$$V = \int\int\int \frac{\rho_1' d\tau_1}{r} + \int\int \frac{\sigma_1' dS_1}{r}. \qquad [195]$$

When V is given, the quantity ρ' is determined at all points where the equation has a definite meaning by $\nabla^2 V = -4\pi\rho'$ and the quantity σ' at all surfaces where the normal derivative of V is discontinuous by the equation $N_1 + N_2 = 4\pi\sigma'$.

Now $\quad D_x(\mu D_x V) + D_y(\mu D_y V) + D_z(\mu D_z V) = -4\pi\rho,$

or $\quad \mu\nabla^2 V + (D_x V \cdot D_x\mu + D_y V \cdot D_y\mu + D_z V \cdot D_z\mu) = -4\pi\rho,$

or $\quad -4\pi\mu\rho' + (D_x V \cdot D_x\mu + D_y V \cdot D_y\mu + D_z V \cdot D_z\mu) = -4\pi\rho, \quad [196]$

and this defines ρ'. In every region where μ is constant $\rho' = \rho/\mu$.

In the most general case of a surface where the normal derivative of V is discontinuous, there is a discontinuity in μ at the surface and a charge, σ, on the surface so that

$$\mu_1 N_1 + \mu_2 N_2 = 4\pi\sigma, \quad N_1 + N_2 = 4\pi\sigma',$$

whence $$\sigma' = \frac{\sigma}{\mu_2} + \frac{N_1(\mu_2 - \mu_1)}{4\pi\mu_2} = \frac{\sigma}{\mu_1} + \frac{N_2(\mu_1 - \mu_2)}{4\pi\mu_1}. \qquad [197]$$

In a particular instance there may be a surface charge with no discontinuity in the dielectric, in which case $\sigma' = \sigma/\mu$; or there may be discontinuity in the dielectric with no real surface charge, in which case

$$\sigma' = N_1(\mu_2 - \mu_1)/4\,\pi\mu_2 = N_2(\mu_1 - \mu_2)/4\,\pi\mu_1.$$

The difference $(\rho' - \rho,\ \sigma' - \sigma)$ between the apparent charge and the intrinsic charge is sometimes called the *induced charge.*

The solving of one or two simple problems will suffice to illustrate the use of the general equations which determine the potential function when the dielectric is not homogeneous.

I. "A condenser consists of two concentric conducting spherical surfaces of radii a and b separated by a dielectric the inductivity of which at a distance, r, from the common centre, O, of the spherical surfaces is $\frac{c+r}{r}\cdot$ The inner plate, of radius a, has a charge E. The outer plate is at potential zero. The potential function in the dielectric is evidently a function of r only; what is its value?"

Since the induction through any closed surface is equal to $4\,\pi$ times the intrinsic charge within, we may imagine a spherical surface drawn in the dielectric with centre at O and radius equal to r and then assert that, if F is the force,

$$4\,\pi r^2\cdot\frac{c+r}{r}\cdot F = 4\,\pi E \text{ so that } F = -\,D_r V = \frac{E}{r(c+r)}$$

and $V = \frac{E}{c}\log\frac{b(r+c)}{r(b+c)}\cdot$ The capacity of the condenser is $c \div \log\frac{b(a+c)}{a(b+c)}\cdot$ The apparent surface density on the inner plate is $\sigma' = E/[4\,\pi a(a+c)]$, the intrinsic surface density is $E/4\,\pi a^2$, and the density of the charge induced at the inner surface of the dielectric is $-\,Ec/[4\,\pi a^2(a+c)]$.

II. "A condenser consists of two large, plane, conducting plates parallel to each other and separated by three slabs, s_1,

s_2, s_3, of dielectric of thickness a, b, and c respectively, and of inductivity 1, μ, and 1. What is the capacity of the condenser per unit area of one of its plates?"

Take the axis of x perpendicular to the faces of the plates with the origin in the first plate, which shall be kept at potential zero. It is evident that the potential function is a function of x only, so that $D_x^2 V = 0$ in each slab of dielectric and V must be of the form $Lx + M$. Denote the functions which give the potential in the three slabs by

$$V_1 = L_1 x + M_1,\ \ V_2 = L_2 x + M_2,\ \ V_3 = L_3 x + M_3.$$

When $x = 0$, $V_1 = 0$. When $x = a$, $-D_x V_1 + \mu D_x V_2 = 0$, and $V_1 = V_2$. When $x = (a + b)$, $-\mu D_x V_2 + D_x V_3 = 0$, and $V_2 = V_3$. We have, therefore, $V_1 = L_1 x$,

$$V_2 = L_1 (x + a\mu - a) / \mu,\ V_3 = L_1 (\mu x + b - b\mu) / \mu.$$

When $x = 0$, $D_x V = -4\pi\sigma = L_1$, and, if $V_3 = 1$ when

$$x = a + b + c, \text{ we get } \sigma = \frac{-\mu}{4\pi(\mu a + \mu c + b)},$$

and this is the capacity per unit area of the first plate.

69. Polarized Distributions. Imagine two homogeneous bodies, P and N, of equal but opposite densities, ρ and $-\rho$, of the same dimensions, and occupying at the same time the same space, in which, of course, the resultant density is zero. If P be moved without rotation through a small distance h, in some direction, there will be a space of no density common to P and N, a space of density ρ where P extends beyond N, and a space of density $-\rho$ where N extends behind P. The thickness of the shell of matter, measured on the exterior normal to the space of no density, is Δn. If, now, h be made to approach zero, and ρ

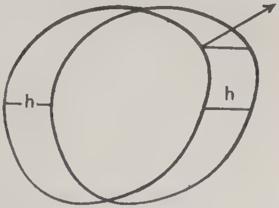

FIG. 49.

be imagined to increase without limit so as to keep the product ρh always equal to a given constant I,

$$\lim_{h \doteq 0} \Delta n / h = \cos(h, n),$$

and we have in the limit merely a superficial distribution, of density $\sigma = I \cos(h, n)$, on the boundary of the space originally occupied in common by P and N. Since the direction of h is fixed in space, and n is an exterior normal, the distribution consists partly of negative matter and partly of positive matter in equal amounts. The surface density is equal to zero at points of contact of the distribution with tangents parallel to the direction of h.

If this distribution be divided up into filaments parallel to h, it is clear that the charges on the ends of every filament are equal and opposite, and that each is equal in amount to qI, where q is the cross-section of the filament in question. It is easy to see from this that if the distribution were placed in a uniform field of force of intensity F, this field would exert upon any such filament of length l a couple of moment $F \cdot \sin(h, F) \cdot qlI$, and upon the whole distribution a couple of moment $F \cdot \sin(h, F) \cdot I$ times the volume of the space enclosed by the distribution. I is, therefore, numerically equal to the moment of the couple, per unit of volume, per unit field perpendicular to the direction of h. The distribution just described is said to be a *uniformly polarized* distribution. I is called the *intensity of the polarization.*

If, for instance, P is a sphere of radius a with centre at O, and if $r^2 \equiv x^2 + y^2 + z^2$, the potential function, $V(x, y, z)$, due to its own mass, has, as we know, the value $2\pi\rho(a^2 - \frac{1}{3}r^2)$ at inside points, and the value $4\pi\rho a^3/3r$ at outside points. After P has been displaced through a distance h parallel to the x axis, the potential function at any point (x, y, z), either in the space common to P and N or outside both, has the value $V(x - h, y, z) - V(x, y, z)$. If the point is within both P and N, the value of this quantity is $2\pi\rho h(2x - h)/3$, but if the point is without both P and

N, the value is $2\pi a^3 \rho h(2x - h)/3r^3$. The limits of these expressions ($4\pi Ix/3$ and $4\pi a^3 Ix/3r^3$) give the values of the potential function within and without a sphere uniformly polarized to intensity I parallel to the x axis. Within the sphere the equipotential surfaces are planes perpendicular to the x axis, the field is uniform, and since $X = -D_x V$, the lines of force are parallel to the negative direction of the x axis. Considerations of symmetry show that the lines of force without the sphere are curves lying in planes through the axis of x. From the expression for V at outside points we learn that if θ is the angle which the radius vector drawn from the origin to any point makes with the x axis, the equipotential surfaces of revolution without the sphere may be considered as generated by plane curves which belong to the family $\cos\theta/r^2 = c$. Curves of this family lying in a plane are cut orthogonally by curves in the same plane which have the equation $r = k \cdot \sin^2\theta$, and this evidently gives the lines of force. Fig. 50 shows the forms of these lines and the direction of the force. It is to be noted that this direction changes abruptly at the surface; on the x axis without the sphere the force is directed from left to right, whereas within the sphere it is

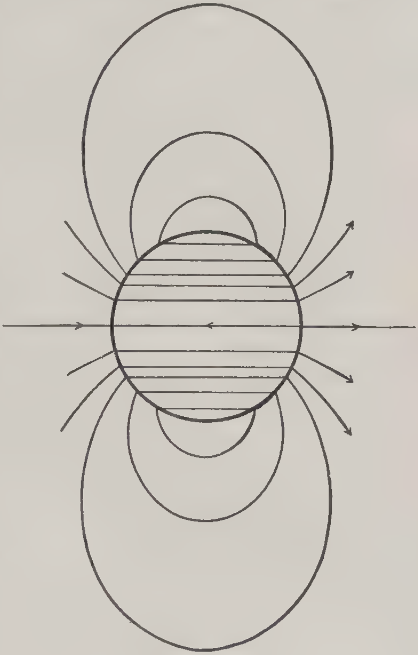

FIG. 50.

directed from right to left. This discontinuity is wholly explained, as a little simple computation will show, by the fact that at any superficial distribution of density σ every tangential component of the force is continuous, but the normal component is discontinuous by $4\pi\sigma$.

The potential function belonging to a uniform field of force of intensity X_0, the lines of which are parallel to the x axis, is $-X_0x$, and if into such a field a sphere of radius a, uniformly polarized to intensity I parallel to the x axis, is brought, and if we define the constant χ by the equation $X_0 = 4\pi I\chi/3$, the potential function, referred to the centre of the sphere as origin, will have the value $4\pi Ix(1-\chi)/3$ at points within the sphere, and the value $4\pi Ix[a^3/3(x^2+y^2+z^2)^{3/2}-\chi/3]$ at outside points. The field within the sphere is now a uniform field of intensity $4\pi I(\chi-1)/3$ directed parallel to the x axis: if $\chi=1$, this force vanishes. The equipotential surfaces of revolution without the sphere could be generated by the revolution about the x axis of a family of curves the equation of which in the xy plane is $4\pi Ix[a^3/3r^3-\chi/3]=c$, where $r^2 \equiv x^2+y^2$. The equation of the family of curves which cut these orthogonally may be written,

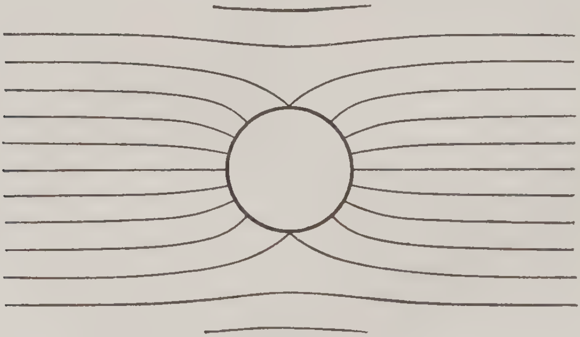

Fig. 51.

$$2\pi Iy^2(2a^3/3r^3+\chi/3)=m,$$

and this represents the lines of force. These lines may be easily plotted for any value of χ, by assuming in succession a series of values of r and computing the corresponding values of y. Figs. 51 and 52 show two characteristic forms which the lines may have. In the first $\chi = +1$, in the second $\chi = -3$. Some slight theoretical interest attaches to the case for which $\chi = -2$, and the reader may care to plot for himself the corresponding curves. He should indicate the direction of the force at various points by arrows.

The value, at inside points, of the potential function due to a homogeneous ellipsoid of density ρ, with axes coincident with the coördinate axes, is given on page 121. If we call this $\rho \cdot \Omega(x, y, z)$, we may write

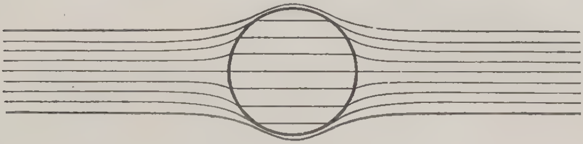

FIG. 52.

$$\Omega(x, y, z) \equiv abc\pi (G_0 - K_0 x^2 - L_0 y^2 - M_0 z^2),$$

where G_0, K_0, L_0, M_0 have the same values at all points of the mass. If, now, we consider an ellipsoidal distribution, uniformly polarized to intensity I, in a direction s, it is easy to see that the value of the potential function within the distribution is

$$-I[D_x\Omega \cdot \cos(x, s) + D_y\Omega \cdot \cos(y, s) + D_z\Omega \cdot \cos(z, s)]$$

or $2\,abc\pi I[K_0 x \cdot \cos(x, s) + L_0 y \cdot \cos(y, s) + M_0 z \cdot \cos(z, s)]$,

and that, if we regard the polarization as a vector and denote its components by A, B, and C, the force components are

$$-2\,\pi abcAK_0, -2\,\pi abcBL_0, -2\,\pi abcCM_0.$$

The field within the distribution is, therefore, uniform, and it has a direction defined by cosines which are to each other as $K_0 \cdot \cos(x, s) : L_0 \cdot \cos(y, s) : M_0 \cdot \cos(z, s)$. It is to be noticed that this direction does not coincide with that of the polarization unless $K_0 = L_0 = M_0$ (so that the ellipsoid is really a sphere), or unless the direction of polarization coincides with that of one of the principal axes of the ellipsoid.

In certain cases the elliptic integral

$$K_0 \equiv \int_0^\infty \frac{ds}{(s + a^2)^{3/2}(s + b^2)^{1/2}(s + c^2)^{1/2}}$$

and the corresponding integrals L_0 and M_0 can be easily evaluated. If, for instance, $a = b = c$, these quantities evidently have the common value $2/3\,a^3$.

If the ellipsoid is a figure of revolution, we may find the values of K_0, L_0, M_0 with the help of the integrals

$$\int \frac{ds}{(s + l^2)(s + m^2)^{1/2}}$$

$$= \frac{1}{(m^2 - l^2)^{1/2}} \cdot \log\left(\frac{(s + m^2)^{1/2} - (m^2 - l^2)^{1/2}}{(s + m^2)^{1/2} + (m^2 - l^2)^{1/2}}\right)$$

or

$$\frac{2}{(l^2 - m^2)^{1/2}} \cdot \tan^{-1}\frac{(s + m^2)^{1/2}}{(l^2 - m^2)^{1/2}},$$

$$\int \frac{ds}{(s + l^2)^2(s + m^2)^{1/2}}$$

$$= \frac{1}{l^2 - m^2}\left(\frac{(s + m^2)^{1/2}}{s + l^2} + \tfrac{1}{2}\int \frac{ds}{(s + l^2)(s + m^2)^{1/2}}\right),$$

$$\int \frac{ds}{(s + l^2)(s + m^2)^{3/2}}$$

$$= \frac{1}{m^2 - l^2}\left(\frac{2}{(s + m^2)^{1/2}} + \int \frac{ds}{(s + l^2)(s + m^2)^{1/2}}\right).$$

In the case of a prolate ellipsoid where $a > b$, $b = c$, and

$$e = \sqrt{a^2 - b^2}/a,$$

$$L_0 = M_0 = \frac{1}{e^2 a^3}\left(\frac{1}{1 - e^2} + \frac{1}{2e} \cdot \log\frac{1 - e}{1 + e}\right),$$

$$K_0 = \frac{2}{e^2 a^3}\left(\frac{1}{2e} \cdot \log\frac{1 + e}{1 - e} - 1\right),$$

and the force components within the ellipsoid are

$$-4\pi A\frac{(1-e^2)}{e^2}\left(\frac{1}{2e}\cdot\log\frac{1+e}{1-e}-1\right),$$

$$-2\pi B\left(\frac{1}{e^2}+\frac{1-e^2}{2e^3}\cdot\log\frac{1-e}{1+e}\right),$$

$$-2\pi C\left(\frac{1}{e^2}+\frac{1-e^2}{2e^3}\cdot\log\frac{1-e}{1+e}\right).$$

If, while b is constant, a be increased without limit, e approaches the limit unity, $(1-e^2)\cdot\log[(1+e)/(1-e)]$ the limit zero, and the ellipsoid becomes an infinitely long cylinder of revolution, for which the force components are $0, -2\pi B, -2\pi C$.

In the case of an oblate ellipsoid where $a < b$, $b = c$, and $e = \sqrt{b^2 - a^2}/b$,

$$K_0 = \frac{2}{e^2b^3}\left(\frac{1}{\sqrt{1-e^2}}-\frac{\sin^{-1}e}{e}\right),$$

$$L_0 = M_0 = \frac{1}{e^2b^3}\left(\frac{\sin^{-1}e}{e}-\sqrt{1-e^2}\right),$$

and the force components within the ellipsoid are

$$-4\pi A\left(\frac{1}{e^2}-\frac{\sqrt{1-e^2}}{e^3}\cdot\sin^{-1}e\right),$$

$$-2\pi B\left(\frac{\sqrt{1-e^2}}{e^3}\cdot\sin^{-1}e-\frac{1-e^2}{e^2}\right),$$

$$-2\pi C\left(\frac{\sqrt{1-e^2}}{e^3}\cdot\sin^{-1}e-\frac{1-e^2}{e^2}\right).$$

If, while b and c are constant, a is made to approach zero, e approaches the limit unity, the limiting values of the force components are $-4\pi A$, 0, 0, and we have the case of a circular disc, in which, if the direction of polarization lies in the plane of the disc, the resultant force is zero.

If the imaginary body P, instead of being of the same density throughout, had consisted of two homogeneous portions of densities ρ_1 and ρ_2, to the left and to the right of their surface of separation, S; if the density of N had been at every point equal and opposite to that of P, and if the limits of $\rho_1 h$ and $\rho_2 h$ had been the constants I_1 and I_2, the resulting surface distribution, on the boundary of the space occupied originally by N and P conjointly, would have had the density $\sigma = I_1 \cos(h, n)$ to the left of the original position of S, and the density $\sigma = I_2 \cos(h, n)$ over the rest of the surface. There would have been on S a surface density $\sigma = I_1 \cos(h, n_1) + I_2 \cos(h, n_2)$, where n_1 and n_2 represent exterior normals to the regions in which P had the densities ρ_1 and ρ_2 respectively. This distribution is therefore equivalent to two distributions uniformly polarized in the direction of h, and laid together so as to have the common surface S.

If, again, the density of P had been given by the expression

$$\rho = \rho_0 \cdot f(x, y, z),$$

where ρ_0 is a constant and f an analytic function of the space coördinates, then, if P had been displaced parallel to the x axis, there would have been, (1) a region common to P and N in which the density would have been

$$\rho_0[f(x-h, y, z) - f(x, y, z)] \text{ or } -\rho_0 h \cdot D_x f + e^2$$

where e is an infinitesimal of the same order as h, (2) a region of density $\rho_0 f(x-h, y, z)$ where P extended beyond N, and (3) a region of density $\rho_0 f(x, y, z)$ where N extended behind P. If the limit of $\rho_0 h$ had been the constant A_0, and if $A_0 \cdot f(x, y, z)$ had been denoted by A, the resulting distribution would have had a surface density $\sigma = A \cos(x, n)$ over the boundary of the space originally occupied by N and P and a volume density $\rho = -D_x A$ inside this boundary. This kind of distribution is called a *non-uniform polarization* of intensity A, the direction of the polarization being that of the x axis. We know from Green's Theorem that the surface integral of $A\cos(x, n)$ taken over any closed surface is equal to the volume integral of $+D_xA$ taken through the space bounded by the surface, so that the whole amount of matter, algebraically considered, in the distribution just discussed is zero.

If such a distribution as this were placed in a uniform field of force of intensity F, perpendicular to the x axis, it would encounter a couple of moment

$$\begin{aligned} M &= F\int\int x\cdot\sigma\cdot dS + F\int\int\int x\cdot\rho\cdot d\tau \\ &= F\int\int x\cdot A\cdot\cos(x, n)\,dS - F\int\int\int x\cdot D_xA\cdot d\tau \\ &= F\int\int\int [D_x(xA) - x\cdot D_xA]\,d\tau \\ &= F\int\int\int A\cdot d\tau. \end{aligned}$$

Here, again, the volume integral of the intensity of the polarization is a measure of the moment of the couple which would be exerted upon the distribution, if it were placed in a uniform field of unit strength perpendicular to the x axis. The intensity of the polarization at any point in a polarized distribution has been called the *moment per unit volume* of the distribution at the point. If a distribution polarized in the manner just described parallel to the x axis were placed in a uniform field (X_0, Y_0, Z_0), not perpendicular to the x axis, it would experience a couple the components of which would be

$$0,\ -Z_0\int\int\int A\cdot d\tau,\ Y_0\int\int\int A\cdot d\tau.$$

If $\rho_0 V(x, y, z)$ is the potential function at (x, y, z) due to P in its original position, the potential function at (x, y, z) due to N and P, after P has been displaced parallel to the axis of x through the distance h, is

$$\rho_0[V(x-h, y, z) - V(x, y, z)],\ \text{or}\ -\rho_0 h\cdot D_xV + e^2,$$

where e is an infinitesimal of the same order as h. As h is decreased and ρ_0 so increased that $\rho_0 h$ is always equal to A_0, the potential function at (x, y, z) due to the resulting distribution becomes $-A_0 \cdot D_x V$. Thus, if P is a sphere of radius a, the density of which is proportional to the distance from its centre, we have $\rho = \rho_0 r$, $V = \pi\rho_0(4a^3 - r^3)/3$ if $r < a$, and $V = \pi\rho_0 a^4/r$ if $r > a$. The polarization in the resulting distribution is $A_0 r$, where A_0 is a constant to be chosen at pleasure; the potential function has the value $\pi A_0 x r$ within the polarized sphere, and $\pi A_0 a^4 x/r^3$ without it; the moment of the sphere is $\pi A_0 a^4$.

Imagine six coincident bodies, P_1, N_1, P_2, N_2, P_3, N_3, of densities $\rho_0 f_1(x, y, z)$, $-\rho_0 f_1(x, y, z)$, $\rho_0 f_2(x, y, z)$, $-\rho_0 f_2(x, y, z)$, $\rho_0 f_3(x, y, z)$, $-\rho_0 f_3(x, y, z)$ respectively. Imagine P_1, P_2, P_3 displaced through distances h parallel, respectively, to the axes of x, y, and z, then imagine h to decrease and ρ_0 to increase in such a way that $\rho_0 h$ is always equal to a constant M. If $Mf_1(x, y, z)$, $Mf_2(x, y, z)$, $Mf_3(x, y, z)$ be denoted by A, B, and C respectively, the resulting distribution has a surface density $\sigma = A\cos(x, n) + B\cos(y, n) + C\cos(z, n)$ on the boundary of the space originally occupied by the six bodies, and a volume density $\rho = -(D_x A + D_y B + D_z C)$ in the region enclosed by this boundary. A, B, C are usually considered to be the components taken parallel to the coördinate axes of a vector, I, so that $\sigma = I\cos(n, I)$ and $\rho = -$ (Divergence of I). The whole amount of matter in the distribution is zero. I is called the *polarization*, and the direction of I at any point is the *direction* at that point *of the polarization*. The lines of the vector I are defined by the equations

$$dx/A = dy/B = dz/C,$$

and are called the *lines of polarization*. If through every point of a curve, s, in a polarized distribution, we draw a line of polarization, we shall get (unless s is itself a line of polarization) a *polarization surface;* if s is closed and the polarization surface tubular, the latter is called a *tube of* *polarization.* The product of the cross-section of a very slender tube of polarization at any point, and the value at that point of I, is sometimes called the *strength* of the tube. The matter in a slender tube of polarization constitutes a *polarized filament.* If the vector I is solenoidal, the distribution is wholly superficial, and the strength of every tube of polarization is constant throughout its length. Uniform polarization is a special case of solenoidal polarization.

It is evident that the generally polarized distribution just mentioned may be regarded as formed by the superposition of three distributions polarized parallel to the axes of x, y, and z respectively, and it is easy to see that a uniformly polarized distribution in a uniform field (X_0, Y_0, Z_0) will be acted on by a couple the components of which are the products of the volume of the distribution and the quantities $BZ_0 - CY_0$, $CX_0 - AZ_0$, $AY_0 - BX_0$.

A short, extremely slender, right prism, uniformly polarized in the direction of its length, forms a simple kind of *polarized element.* If $2l$ is the length of such an element, q its cross-section, and I the intensity of its polarization, $2qlI$ may be called the *moment* of the element, for it represents the moment of the couple which would act upon the element if it were placed perpendicularly across the lines of a unit field. This product of the volume of the element and I we may denote by M. We know that the field of force due to the element is mathematically accounted for by a superficial negative charge, $-qI$, on one end of the prism and an equal positive charge on the other end. Let Q be any point distant r_1 from the negative end and r_2 from the positive end of the axis of the prism, and r from its centre. Let (r, I) be the angle between the direction of polarization and the line drawn from the centre of the axis to Q; then, since

$$r_1^2 = r^2 + l^2 + 2\,rl \cdot \cos(r, I) \text{ and } r_2^2 = r^2 + l^2 - 2\,rl \cdot \cos(r, I),$$

the value at Q of the potential function due to the element is $qI(1/r_2 - 1/r_1)$ or

$$qI(r_1^2 - r_2^2)/r_1r_2(r_1 + r_2) \quad \text{or} \quad 2\,rM \cdot \cos(r, I)/r_1r_2(r_1 + r_2).$$

The limit, $M \cdot \cos(r, I)/r^2$, of the expression just found is called the potential function due to a uniformly *polarized element*, or to a *space doublet*. It will appear from the work which follows that a similar result might have been obtained from the use of a generally polarized element of any form. The lines of force due to a polarized element are shown in Fig. 53; they are the same as the external lines of force in Fig. 50.

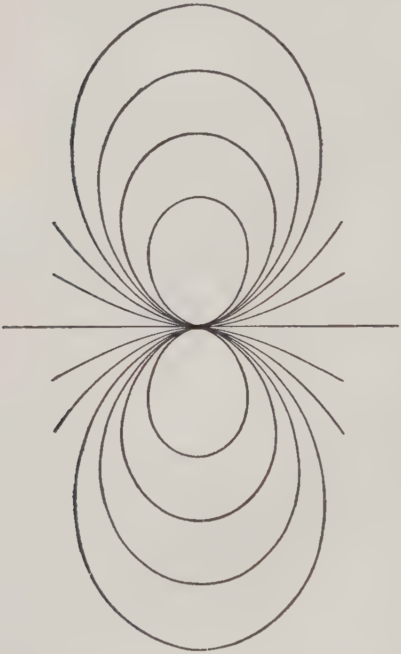

Fig. 53.

Before we attempt to find an expression for the potential function due to a generally polarized finite distribution, it is well to notice that if the vector I is discontinuous at any surfaces, the distribution may be considered as made up of a number of continuously polarized portions abutting at these surfaces: we may confine our attention, therefore, to continuously polarized distributions. If a given distribution of this kind has the volume density ρ' in a region T' and a surface density σ' on the closed surface S' which bounds T', the potential function at the point (x, y, z) due to the distribution is

$$V=\iint\frac{\sigma' dS'}{r}+\iiint\frac{\rho' d\tau'}{r},$$

where r is the distance *from* the point (x', y', z') in the volume or surface element *to* the point (x, y, z). If I' is the value of the polarization at (x', y', z'), and if we substitute the values of ρ' and σ' in terms of I' and its components A', B', C', we have

$$V=\iint\frac{[A'\cos(x', n)+B'\cos(y', n)+C'\cos(z', n)]}{r}dS'$$
$$-\iiint\frac{D_{x'}A'+D_{y'}B'+D_{z'}C'}{r}d\tau'.$$

Since $$D_{x'}\left(\frac{A'}{r}\right)\equiv\frac{D_{x'}A'}{r}-\frac{A'\cdot D_{x'}r}{r^2},$$

we may write

$$-\iiint\frac{D_{x'}A'}{r}d\tau'$$
$$=-\iiint\frac{A'\cdot D_{x'}r\cdot d\tau'}{r^2}-\iiint D_{x'}\left(\frac{A'}{r}\right)d\tau'$$
$$=-\iiint\frac{A'\cdot D_{x'}r\cdot d\tau'}{r^2}-\iint\frac{A'}{r}\cos(x', n)\,dS',$$

with similar expressions for the other terms of the triple integral, all the double integrals are cancelled, and we have

$$V=-\iiint\frac{A'\cdot D_{x'}r+B'\cdot D_{y'}r+C'\cdot D_{z'}r}{r^2}d\tau'$$
$$=\iiint\frac{A'(x-x')+B'(y-y')+C'(z-z')}{r^3}d\tau'$$
$$=\iiint\frac{A'\cos(x', r)+B'\cos(y', r)+C'\cos(z', r)}{r^2}d\tau'$$
$$=\iiint\frac{I'\cos(r, I')}{r^2}d\tau'.$$

The quantity under the integral signs in this last expression is, as we have just seen, the potential function due to an element at (x', y', z'), in which the polarization is I'.

If the polarization is solenoidal, the volume integral of ρ' is equal to zero and a surface integral alone remains.

We have seen that a polarized distribution is completely defined when the form of its boundary, S, and the values of the components of the vector I within it are known, and that its potential function has the same value (at least at outside points) as that due to an ordinary distribution of matter made up of a certain volume distribution within S and a certain superficial distribution on S. What we usually call a polarized distribution is supposed to be quite different, however, in its physical nature from this ordinary distribution, which may be said to be mathematically equivalent to it. A simple illustration will make the character of this difference clear.

If a number of small cubes, all uniformly polarized parallel to one edge, with common intensity I, were placed together, with their directions of polarization parallel, to form a larger cube, P, superficial distributions of equal and opposite densities would come in contact, and the resulting distribution would appear to consist only of a positive charge uniformly spread on one face of the larger cube and an equal negative charge spread uniformly on the opposite face. That is, the potential function, at outside points, due to P, would be the same as that due to an indifferent body, P', of the same dimensions as P, charged with a superficial distribution of density $+I$ on one face and a superficial distribution of density $-I$ on the opposite face. If, however, we define the force at a point within a distribution to be the force which would urge a unit mass concentrated at the point, if an infinitesimal cavity were excavated at the point to allow of its introduction, the intensity of the force at a given point within P' might be very different from that of the force at the corresponding point in P. The first would be due merely to the surface charges already mentioned, whatever the shape of the cavity, while if an excavation were made in P by removing one of the very small uniformly polarized cubes of which it is made up, the surface charges on the adjacent cubes would appear, and, however small the cavity might be, these would be found to modify the force very appreciably. We must regard such polarized distributions as occur in nature as made up of *polarized molecules*, so that if any portion be broken off, across the lines of polarization, from a body in which the polarization is defined by the vector I, each portion is a polarized distribution defined by the same vector as before at every point, so that a surface distribution appears on each of the new faces formed by the fracture. Every magnet appears to be a polarized distribution of magnetic matter, and problems in magnetism, as the reader who has some knowledge of magnetic phenomena will see, can be conveniently attacked by the analysis of this section.

If into a field of electric force a conductor or a mass of dielectric different in nature from that which it displaces be introduced, the field becomes changed in a manner completely explainable on the assumption that the conductor, or the dielectric, has become electrically polarized, and that the surrounding dielectrics are now and were polarized. Indeed, results of experiment compel us to assume that space which seems to be empty of ponderable matter is still occupied by a medium, the ether, capable of transmitting electrical forces. We must assume, also, that every medium with which we are acquainted, whether it be solid, liquid, gaseous, or ethereal, is susceptible to electrical and to magnetic forces, so that if a mass of any isotropic medium be placed in a field of electric (or magnetic) force, it becomes polarized by induction in such a manner that the direction of the electric (or magnetic) polarization coincides at every point with the direction of the resultant electric (or magnetic) force due to all the apparent electrical (or magnetic) matter in the universe, including that which belongs to the polarization of the medium itself. The ratio of the intensity of the polarization induced at any point of a medium by the resultant force at the point is called the *susceptibility* of the medium at the point under the given circumstances. Every medium has both an electrical susceptibility and a magnetic susceptibility, and these may be represented by very different numbers. The susceptibility of a medium to *magnetic* influences often depends upon the intensity of the inducing force; we may consider, however, that, if a medium is homogeneous, its *electrical* susceptibility (k) has the same value throughout. A medium in a field of force may have an *intrinsic polarization* as well as the polarization induced in it by the field. A steel magnet in the earth's field illustrates this possibility.

A given region may be at once a field of magnetic force and a field of electric force, so that any medium, when placed in this region, becomes both magnetically and electrically polarized. Since the two polarizations are similar, we need speak in what follows only of one, if we keep in mind the fact that two quite independent polarizations may coexist. We shall represent susceptibility by k and inductivity by μ, with the understanding that different numerical values must be assigned in general to these quantities according as we are dealing with electrical or magnetic phenomena.

In the most general case of either electrical or magnetic polarization we may imagine that an isotropic medium has (1) an intrinsic volume charge of density ρ_0, where ρ_0 is a scalar point function, (2) superficial intrinsic charges over certain surfaces, and (3) an intrinsic polarization I_0, with components A_0, B_0, C_0, which may or may not be everywhere continuous. In addition to this it has (4) an induced polarization which, as we have seen, has the direction of the resultant force coming from all the apparent charges in existence, including those which come from the intrinsic and induced polarization of the medium itself. We may assume for our present purposes that k depends upon the character of the medium, but not upon the intensity of the resultant force (X, Y, Z). The whole apparent volume density, according to the statements just made, is, in the general case,

$$\rho = \rho_0 - [D_x A_0 + D_y B_0 + D_z C_0] \\ - [D_x(kX) + D_y(kY) + D_z(kZ)],$$

and this is equal, according to Poisson's Equation, to

$$-\nabla^2 V/4\pi, \text{ or to } (D_x X + D_y Y + D_z Z)/4\pi.$$

If we denote $1 + 4\pi k$ by μ, this equation may be written in several interesting forms, and may be regarded as a generalized Poisson's Equation.

$$D_x(\mu X) + D_y(\mu Y) + D_z(\mu Z) \\ = 4\pi[\rho_0 - (D_x A_0 + D_y B_0 + D_z C_0)],$$

$$D_x(\mu X + 4\pi A_0) + D_y(\mu Y + 4\pi B_0) + D_z(\mu Z + 4\pi C_0) = 4\pi\rho_0,$$

$$D_x[X + 4\pi(kX + A_0)] + D_y[Y + 4\pi(kY + B_0)] \\ + D_z[Z + 4\pi(kZ + C_0)] = 4\pi\rho_0.$$

The vector, the components of which are $(\mu X + 4\pi A_0)$, $(\mu Y + 4\pi B_0)$, $(\mu Z + 4\pi C_0)$, or, what is the same thing, $(X + 4\pi A)$, $(Y + 4\pi B)$, $(Z + 4\pi C)$, where A, B, C are the components, $(kX + A_0)$, $(kY + B_0)$, $(kZ + C_0)$, of the resultant polarization arising from the superposition of the intrinsic and the induced polarizations, is called the *generalized induction.* At a charged surface, the sum of the normal components of the generalized induction, pointing away from the surface on both sides, is evidently equal to $4\pi\sigma_0$. The sum of the normal components of the force, pointing away from the surface on both sides, is equal to $4\pi\sigma$, while every tangential component of the force is continuous at the surface.

If in a homogeneous medium incapable of being polarized inductively, where there is an intrinsic polarization I_0 (with components A_0, B_0, C_0), but no intrinsic body or surface charges not accounted for by the polarization, the polarization within the medium coincides everywhere in direction with the force due to all the active matter in existence, including the polarization masses, and if the intensities of the polarization and force have everywhere the constant ratio λ, the divergence of the polarization is evidently equal to λ times the divergence of the force, and Poisson's Equation becomes

$$D_xX + D_yY + D_zZ = -4\pi\lambda(D_xX + D_yY + D_zZ),$$

so that the force and the polarization must be solenoidal. The converse of this theorem is evidently not true.

Inductive bodies which are incapable of being intrinsically polarized are sometimes said to be electrically or magnetically *soft.* Most isotropic substances seem to be electrically soft. Bodies which can be intrinsically polarized, but which are assumed to be incapable of being polarized by induction, are sometimes said to be electrically or magnetically *hard.* No absolutely hard media are known to exist, but the magnetic susceptibilities of some permanent magnets are comparatively small.

The generalized Poisson's Equation becomes

$$D_x(\mu X) + D_y(\mu Y) + D_z(\mu Z) = 4\pi\rho_0$$

in the case of a body which has no intrinsic polarization, and the generalized induction becomes the simple vector $(\mu X, \mu Y, \mu Z)$ discussed in the last section. This vector coincides in direction with the resultant force at every point. At a charged surface which also separates two media of different inductivities, the tangential components of the force are continuous, but the product of the tangential component of the force and the inductivity is clearly not continuous. The normal component of the force is discontinuous by 4π times the apparent density of the charge on the surface, while the normal components of the induction are discontinuous by 4π times the density of the intrinsic charge on the surface. In general, therefore, in soft media, neither the tangential nor the normal components of the induction are continuous, but the directions of the force and induction coincide with each other close to the surface on both sides of it.

If k is independent of the intensity of the resultant force, the volume density, $-[D_x(kX) + D_y(kY) + D_z(kZ)]$, due to the induced polarization in a *homogeneous* medium, may be written $-k\,(D_xX + D_yY + D_zZ)$, or $k \cdot \nabla^2 V$, and this vanishes at all points where there are no intrinsic body charges. We must, therefore, consider homogeneous dielectrics about charged bodies to be solenoidally polarized.

If a mass, M, of a soft homogeneous medium of inductivity μ_1, be introduced into a given field of force in an indefinitely extended homogeneous medium of inductivity μ_2 which contains no "real" charges at a finite distance from M, the two media become solenoidally polarized, there is an "apparent" charge, e', at the surface S, of M, and the potential function V is now the sum of the given potential function V_0, which defined the field when the place occupied by M was filled with medium of inductivity μ_2, and the potential function V', which might be computed from the expression Limit $\Sigma(de'/r)$, since it is equal to the potential function in a medium of unit inductivity due to a real charge, e'. If n_1 and n_2 are normals to S drawn respectively into and out of M, we have at every point of S,

$$\mu_1 \cdot D_{n_1} V + \mu_2 \cdot D_{n_2} V = 0,\ D_{n_1} V_0 + D_{n_2} V_0 = 0,$$

and, if $\lambda \equiv \mu_1/\mu_2$,

$$\lambda \cdot D_{n_1} V' + D_{n_2} V' + (\lambda - 1)\, D_{n_1} V_0 = 0,$$

in which λ is positive, and, since V_0 is given, the last term of the first member is a given function, $f(x, y, z)$. V' is continuous at S, it is harmonic within and without S, and it vanishes canonically at infinity; it is easy to show that all these conditions determine V' (and, therefore, e') uniquely. For if we assume that two different functions may satisfy the conditions, and if we denote their difference by u, u must vanish canonically at infinity, be continuous at S, and satisfy Laplace's

Equation within and without S. On S, $\mu_1 \cdot D_{n_1}u + \mu_2 \cdot D_{n_2}u = 0$. If, now, we apply [149] to u, choosing for λ of that equation the value μ_1 in the space within S and the value μ_2 in the space without S, it is evident that u must have everywhere the value zero. It is to be noted that changes in μ_1 and μ_2, which did not affect their ratio, would not affect V'.

If the strength of the given field had been greater in the constant ratio m than it was, the potential function V'', due to the apparent charge on S, would have been larger in the same ratio, for [149] shows that the difference between the two functions V'' and mV' (both of which vanish canonically at infinity, are continuous at S, are harmonic within and without S, and at S satisfy an equation of the form

$$\lambda \cdot D_{n_1}V'' + D_{n_2}V'' + m \cdot f(x, y, z) = 0),$$

is identically zero.

If, when a hard body, M, solenoidally polarized intrinsically, is placed in a field of force in a homogeneous soft medium of inductivity unity, the directions of the polarization and of the resultant force are found to coincide within M, and, if the ratio of the intensities of these vectors is equal at every point of M to the constant k, the potential function within and without M is the same as if M were a homogeneous, perfectly soft medium of susceptibility k and inductivity $1 + 4\pi k$, polarized inductively by the original field. To prove this we have only to compare the properties of the potential functions in the two cases. Let S be the bounding surface of M, let n_1 and n_2 represent respectively interior and exterior normals to S, and let V_0 be the potential function due to the original field; then V_0 is harmonic within S, and on S, where V_0 is continuous, $D_{n_1}V_0 + D_{n_2}V_0 = 0$. Let I' be the polarization in the hard body M, and I'' the polarization which would be induced if S were filled by a soft medium of inductivity $1 + 4\pi k$, and let V' and V'' be the corresponding potential functions. The components of I' and I'' in any direction at any point within S are respectively equal to $-k$ times the derivatives, at the point, in the given direction, of $V_0 + V'$ and $V_0 + V''$. The density σ' of the real charge of S in the case of the hard body is $-I' \cdot \cos(n_1, I')$ or $k \cdot D_{n_1}(V_0 + V')$ and the density σ'' of the charge which would be induced on S, if M were displaced by the soft medium, is $-I'' \cdot \cos(n_1, I'')$ or $k \cdot D_{n_1}(V_0 + V'')$. On S,

$$D_{n_1}(V_0 + V') + D_{n_2}(V_0 + V') = -4\pi k \cdot D_{n_1}(V_0 + V'),$$

and $$(1 + 4\pi k) \cdot D_{n_1}(V_0 + V'') + D_{n_2}(V_0 + V'') = 0,$$

or $$(1 + 4\pi k) \cdot D_{n_1} V' + D_{n_2} V' = -4\pi k \cdot D_{n_1} V_0,$$

and $$(1 + 4\pi k) \cdot D_{n_1} V'' + D_{n_2} V'' = -4\pi k \cdot D_{n_1} V_0.$$

If, now, $u \equiv V' - V''$, u is harmonic within and without S, it vanishes canonically at infinity, and it is continuous on S, where $(1 + 4\pi k) \cdot D_{n_1}u + D_{n_2}u = 0$. It is easy to prove, with the help of [149], that under these circumstances u must be identically equal to zero, so that V' and V'' are identically equal.

We know from work done earlier in this section that when a uniformly polarized sphere is placed in a uniform field of force of intensity X_0, so that the direction of the polarization and this field coincide, the resultant field within the sphere is a uniform field of intensity $X_0 - 4\pi I/3$ in the direction of the polarization, and that the ratio of the polarization and the force is the constant $k = 3I/(3X_0 - 4\pi I)$, or $3/4\pi(\chi - 1)$, so that $I = 3X_0[(1 + 4\pi k) - 1]/4\pi[(1 + 4\pi k) + 2]$. We infer from this that if a sphere of soft medium of inductivity μ were placed in a uniform field of force of intensity X_0 in a soft medium of unit inductivity, the sphere would become uniformly polarized to intensity $3X_0(\mu - 1)/4\pi(\mu + 2)$ and that the uniform field inside the sphere would have the direction of the outside field and the intensity $3X_0/(\mu + 2)$. Since in such a case as this the ratio of the inductivities of the inner and outer media is alone important, we may say that, if a soft sphere of inductivity μ_1 were placed in a uniform field of intensity X_0 in a soft medium of inductivity μ_2, the sphere would become uniformly polarized to intensity

$$3\,X_0\,(\mu_1/\mu_2-1)\,/\,4\,\pi\,(\mu_1/\mu_2+2),$$

or

$$3\,X_0\,(\mu_1-\mu_2)\,/\,4\,\pi\,(\mu_1+2\,\mu_2),$$

and that the intensity of the uniform resultant field within the sphere would be $3X_0/(\mu_1/\mu_2+2)$ or $3\,\mu_2X_0/(\mu_1+2\,\mu_2)$. That part, $X_0\,(\mu_1-\mu_2)\,/\,(\mu_1+2\,\mu_2)$, of the field within the sphere which is due to the polarization alone, is negative, if $\mu_1>\mu_2$, and is then called the *self-depolarizing* force.

If we note that in the analysis accompanying Figs. 51 and 52, χ was defined by the equation $X_0=4\,\pi I\chi/3$ and that consequently $\chi=(\mu_1/\mu_2+2)\,/\,(\mu_1/\mu_2-1)$, we may apply to our present subject all the work there done. These figures represent the lines of force for the cases $\mu_1/\mu_2=\infty$ and $\mu_1/\mu_2=\frac{1}{4}$ respectively; the first corresponds to a perfect conductor in a uniform electric field, or (approximately) to a sphere of very soft iron in a uniform magnetic field in air. The theory of the polarization by induction of a soft sphere in a uniform field was first given by Lord Kelvin, and this theory, with diagrams for $\mu_1/\mu_2=2.8$ and $\mu_1/\mu_2=0.48$, may be found in his *Reprint of Papers on Electrostatics and Magnetism.* Very interesting figures, drawn for equal intervals of the function m and corresponding to $\mu_1/\mu_2=3$, $\mu_1/\mu_2=\infty$, are given on pages 373 and 374 in Professor Webster's *Theory of Electricity and Magnetism.*

If an ellipsoid, made of inductively hard material and uniformly polarized $[I=(A,\,B,\,C)]$, be placed in a uniform field of force $(X_0,\ Y_0,\ Z_0)$, the resultant field within the ellipsoid will evidently be uniform and its components will be

$$X_0-2\,\pi abcAK_0,\ Y_0-2\,\pi abcBL_0,\ Z_0-2\,\pi abcCM_0\,;$$

the directions of the polarization and the field will not agree, however, unless these components are as $A:B:C$, which will be the case if

$$A = kX_0/(1 + 2\,\pi abckK_0),$$
$$B = kY_0/(1 + 2\,\pi abckL_0),$$
$$C = kZ_0/(1 + 2\,\pi abckM_0),$$

where k is any constant. If the intrinsic polarization of the ellipsoid satisfies these conditions, the ratio of the intensities of the polarization and the field is k. We infer from this that if a homogeneous ellipsoid of inductively soft material of susceptibility k be placed in a uniform field of force in a medium of unit inductivity, the ellipsoid will become uniformly polarized and that the components of the polarization will have the values just given. The resultant field within the ellipsoid will have the components

$$X_0/(1 + 2\,\pi abckK_0),\ Y_0/(1 + 2\,\pi abckL_0),\ Z_0/(1 + 2\,\pi abckM_0)$$

and the self-depolarizing force the components

$$-2\,\pi abcK_0A,\ -2\,\pi abcL_0B,\ -2\,\pi abcM_0C.$$

These results may be expressed in terms of the inductivity μ of the ellipsoid by writing $(\mu - 1)/4\,\pi$ for k, and if, then, the ratio μ_1/μ_2 be substituted for μ, the formulas will correspond to the case of a soft homogeneous ellipsoid of inductivity μ_1 in a field of force in a homogeneous medium of inductivity μ_2.

If we remember the expression already found for the moment of the couple which a uniform field exerts upon a uniformly polarized distribution in it, we shall see that in the present case the components of this couple are

$$4\,\pi abc(BZ_0 - CY_0)/3,\ 4\,\pi abc(CX_0 - AZ_0)/3,$$

and $$4\,\pi abc(AY_0 - BX_0)/3,$$

or

$$8\,\pi^2a^2b^2c^2k^2Y_0Z_0(M_0 - L_0)/3(1 + 2\,\pi abckM_0)(1 + 2\,\pi abckL_0),$$
$$8\,\pi^2a^2b^2c^2k^2Z_0X_0(K_0 - M_0)/3(1 + 2\,\pi abckK_0)(1 + 2\,\pi abckM_0),$$
$$8\,\pi^2a^2b^2c^2k^2X_0Y_0(L_0 - K_0)/3(1 + 2\,\pi abckL_0)(1 + 2\,\pi abckK_0);$$

and it is to be noted that, according to the reasoning of page 122, if $a > b > c$, $K_0 < L_0 < M_0$. If the lines of the field in the air make an acute angle with the plane of xz and are perpendicular to the z axis, $Z_0 = 0$ and X_0 and Y_0 are positive, the axis of the couple is the axis of z, the moment is positive (even for such negative values of k as occur in nature), and the ellipsoid tends to turn so that its long axis shall have the direction of the field.

For perfectly hard, intrinsically polarized bodies the generalized Poisson's Equation becomes

$$D_x(X + 4\pi A_0) + D_y(Y + 4\pi B_0) + D_z(Z + 4\pi C_0) = 4\pi\rho_0,$$

and if $\rho_0 = 0$, as in the case of a hard magnet, the induction

$$(X + 4\pi A_0,\quad Y + 4\pi B_0,\quad Z + 4\pi C_0)$$

is solenoidal. Unless the intrinsic polarization happens to have the same direction as the resultant force, or vanishes, the lines of force and the lines of induction do not coincide. In some cases, the directions of the force and of the polarization are exactly opposed, and the lines of force and of induction are opposite in direction. Outside a hard magnet, where the intrinsic polarization is nothing, the lines of force and of induction are identical. At the surface of the magnet, where there is no intrinsic charge except that which belongs to the polarization, the normal component of the force is discontinuous, while the normal component of the induction is continuous. It is convenient, therefore, to regard the lines of induction as closed curves. The lines in Fig. 50 represent both lines of force and lines of induction, but it is to be noticed that inside the sphere, where $X = -4\pi I/3$, the direction of the lines of force is from right to left, while the direction of the lines of induction (since $I = +8\pi I/3$) is from left to right. The straight line of force through the centre of the sphere is discontinuous in direction at the surface, while the corresponding line of induction is continuous. The tangential components of the force are continuous at the surface of a magnet, those of the induction discontinuous. The normal component of the induction just within the surface is

$$(X + 4\pi A_0)\cos(x, n) + (Y + 4\pi B_0)\cos(y, n) \\ + (Z + 4\pi C_0)\cos(z, n),$$

or the normal component of the force plus 4π times the density on the surface belonging to the intrinsic polarization.

If we make a small cavity inside a generally polarized hard body, the force at any point of the cavity is the original value of F (that is, the negative of the gradient of the potential function at the point), minus the contribution due to the volume charge removed, plus the force due to the new surface charges which appear on the walls of the cavity. If the volume of the cavity be made smaller and smaller, the contribution due to the volume charge removed can be made as small as we like, while the effect of the surface charges may remain finite. Let the cavity be of the form of a piece of a slender tube of polarization lying between two near orthogonal surfaces. In this case there will be surface charges on the ends of the cavity, but none on the side walls. These charges, of density $\pm I$, will be such as to drive a particle of positive matter in the centre of the cavity to that end of the cavity towards which the polarization is directed. If we reflect that a surface distribution of finite size, which has at the point P the density σ, repels a unit point charge infinitely near P with a force of $2\pi\sigma$, but that an element of the surface at P, infinitely small with respect to the distance of a point charge from P, has no perceptible effect upon this point charge, it will be easy to see that, if the cross-section of the cavity is infinitely small compared with its length, the force due to the surface charges on the ends approaches zero as the whole cavity is made smaller, and the force at the centre of the cavity is F. If, however, the length of the cavity is infinitely small compared with its cross-section, the force due to the charges on the ends is $4\pi\sigma$, or $4\pi I$, so that the whole force is $F + 4\pi I$, or the induction at the point before the cavity was cut. If the infinitely small cavity were spherical, the force at its centre would be $F + \frac{4}{3}\pi I$.

It is to be carefully noticed that we have no means of determining an absolute inductivity for any medium, but only the ratio of the inductivity of the medium to the inductivity of some other medium taken as a standard. The unit quantity of electricity is defined to be that quantity which concentrated at a point at a distance of one centimetre from an equal quantity would repel it with a force of one dyne, *when the dielectric is the ether.* In any other homogeneous dielectric of inductivity μ times that of the ether, e of these units of electricity concentrated at each of two points distant r centimetres from each other would repel each other with a force of $e^2/\mu r^2$ dynes, so that, if this medium had been used as a standard, the unit of electricity would have been larger in the ratio of $\sqrt{\mu}$ to 1 than it now is. If a charged conductor is enveloped by an infinite, homogeneous dielectric, we may assume the apparent charge on it to be its real charge and neglect the polarization of the dielectric (and we do this when the dielectric is the standard substance which we assume to have unit inductivity, and hence no susceptibility); or we may suppose the dielectric to be polarized, and consider the apparent charge to be the algebraic sum of the real charge on the conductor and the charge belonging to the polarization induced on the dielectric. This we do when the dielectric is not the standard substance, assigning to it a susceptibility based on that of the standard substance which is the zero of our scale. A simple illustration will tend to make the rather complex relations which would attend a change in the choice of a standard substance more intelligible.

A spherical conductor of radius a has a charge, E, and is surrounded by three spherical shells of homogeneous dielectric, concentric with it, the last reaching to infinity. The radii of the spherical surfaces which separate the first (inner) dielectric from the second, and the second from the third, are b and c respectively, and the inductivities of the dielectrics, referred to some standard substance, are μ_1, μ_2, and μ_3, or $1 + 4\pi k_1$, $1 + 4\pi k_2$, and $1 + 4\pi k_3$. If we apply Gauss's Theorem successively to spherical surfaces concentric with the conductor and lying in the first, second, and third media, we learn that the force $(-D_r V)$ at a distance r from the centre has, in the three media, the values $E/\mu_1 r^2$, $E/\mu_2 r^2$, $E/\mu_3 r^2$. The conductor acts like a medium of infinite susceptibility. The induced polarizations in the three media are directed radially outward, and their intensities are $k_1 E/\mu_1 r^2$, $k_2 E/\mu_2 r^2$, $k_3 E/\mu_3 r^2$. The densities of the apparent charges at the surface S_1 of the conductor and at the surfaces of separation S_2, S_3 of the dielectrics, regarded as manifestations of the polarizations, are

$$\sigma'_1 = E/4\pi a^2 - k_1 E/\mu_1 a^2 = E/4\pi a^2 \mu_1 \text{ on } S_1,$$

$$\sigma'_2 = k_1 E/\mu_1 b^2 - k_2 E/\mu_2 b^2 = E(\mu_1 - \mu_2)/4\pi b^2 \mu_1 \mu_2 \text{ on } S_2,$$

and $$\sigma'_3 = k_2 E/\mu_2 c^2 - k_3 E/\mu_3 c^2 = E(\mu_2 - \mu_3)/4\pi c^2 \mu_2 \mu_3 \text{ on } S_3.$$

These same densities might also be found by the aid of the ordinary characteristic equation of the potential function at an apparently charged surface, $D_{n'} V + D_{n''} V = -4\pi\sigma'$.

If instead of using the old standard we make the outer dielectric of this problem the standard, the unit of electrical quantity will be larger in the ratio of $\sqrt{\mu_3}$ to 1 than it was before, and the old charge on the condenser will be $E/\sqrt{\mu_3}$ expressed in the new units. The strength of a field at any point being the force in dynes which would be experienced by a unit of positive matter placed at the point, the number which expresses the strength of a given field in the new units is $\sqrt{\mu_3}$ times the number which expresses the strength of the same field in the old units. The inductivities of the three dielectrics are now μ_1/μ_3, μ_2/μ_3, 1, and their susceptibilities are

$$(\mu_1 - \mu_3)/4\pi\mu_3 \text{ or } (k_1 - k_3)/(1 + 4\pi k_3),$$

$$(\mu_2 - \mu_3)/4\pi\mu_3 \text{ or } (k_2 - k_3)/(1 + 4\pi k_3), \text{ and } 0.$$

The strengths of the fields in the three dielectrics are

$$E\sqrt{\mu_3}/\mu_1 r^2,\ \ E\sqrt{\mu_3}/\mu_2 r^2,\ \ E\sqrt{\mu_3}/\mu_3 r^2;$$

the intensities of the polarizations are

$$E(\mu_1-\mu_3)/4\,\pi\sqrt{\mu_3}\,\mu_1 r^2,\ \ E(\mu_2-\mu_3)/4\,\pi\sqrt{\mu_3}\,\mu_2 r^2,\ 0;$$

and the apparent charges on S_1, S_2, S_3 have the densities

$$E\sqrt{\mu_3}/4\,\pi a^2\mu_1,\ \ E(\mu_1-\mu_2)\sqrt{\mu_3}/4\,\pi b^2\mu_1\mu_2,$$

and

$$E(\mu_2-\mu_3)\sqrt{\mu_3}/4\,\pi c^2\mu_2\mu_3.$$

The sum of the apparent charges on S_1, S_2, S_3 is now $E/\sqrt{\mu_3}$, the real charge on the conductor expressed in the new units, and the sum of the induced charges is zero. In the case first treated, where the outer medium was supposed to be polarizable, the sum of the apparent charges on S_1, S_2, S_3 was E/μ_3, and this, being expressed in the old units, is equivalent to $E/\sqrt{\mu_3}$ in the new. The sum of the induced charges was the difference between E/μ_3 and E or $E(1-\mu_3)/\mu_3$; in this case, however, we must imagine the outer surface "at infinity" of the outer medium to have an induced charge in total amount equal to the integral of the normal component of the polarization (k_3E/μ_3r^2) over the surface, or $4\,\pi k_3E/\mu_3$, and this is equal to $E(\mu_3-1)/\mu_3$, so that here, again, the whole amount of the induced charge is, of course, zero. It is to be noted that this finite charge at infinity does not affect the electrical field in any way. We have seen that when the outer medium is taken as a standard the inner medium has a susceptibility $(\mu_1-\mu_3)/4\,\pi\mu_3$, and this is sometimes called the susceptibility of a medium of inductivity μ_1 with respect to a medium of inductivity μ_3. No medium has yet been found to be less electrically susceptible than the ether. Some bodies are less magnetically susceptible than the ether, so that their susceptibilities are negative on the usual scale. These bodies are said to be diamagnetic.

If a body of inductivity μ_1, bounded by the surface S, is placed in a large mass of a medium of inductivity μ_2, the outer surface of which is so far removed from the place of observation that the apparent charge on it contributes little to the field of force, the fact that the outer medium is really polarized may be lost sight of; and if we attribute the apparent charge on S wholly to the polarization of the inner medium, instead of regarding it as the difference between the charge of one sign due to the polarization of the inner medium, and the charge of the opposite sign due to the polarization of the outer medium, the apparent susceptibility of this medium will be $(\mu_1 - \mu_2)/4\pi\mu_2$. If μ_2 is greater than μ_1, this will be negative and the inner medium will seem to be polarized in a direction opposite to that of the force.

If in any given case the direction of the vector I is everywhere perpendicular to the direction of its curl, it is possible to cut a polarized distribution by a set of surfaces, $u = c$, everywhere normal to the line of polarization. If surfaces of this family be drawn for small constant differences, Δu, of the scalar point function u, the distribution will be divided into shells, each of which is polarized normally to its surface. If Δn is the thickness of one of these shells at a given point and I_0 the average intensity of polarization on a line of polarization drawn through the shell at the point, $I_0 \Delta n$ is called the *strength* of the shell at the point. Since $D_n u = h_u$, the value of the gradient of u, the strength of a shell of infinitesimal thickness can be written $I \cdot du / h_u$. A shell is said to be *simple* if I/h_u has the same numerical value all over it; otherwise the shell is said to be *complex*.

If A, B, C are the intensities of the components of the vector I, the fact that the lines of I coincide with the normals to the surface $u = c$ gives the scalar equations

$$A/I = D_x u / h_u, \quad B/I = D_y u / h_u, \quad C/I = D_z u / h_u;$$

and with the help of these the vector

$$[D_y C - D_z B,\ D_z A - D_x C,\ D_x B - D_y A],$$

which is the curl of I, may be written in the form

$$[D_z u \cdot D_y(I/h_u) - D_y u \cdot D_z(I/h_u),$$
$$D_x u \cdot D_z(I/h_u) - D_z u \cdot D_x(I/h_u),$$
$$D_y u \cdot D_x(I/h_u) - D_x u \cdot D_y(I/h_u)].$$

If, now, the scalar quantity I/h_u has the same numerical value over every surface of constant u, it must be, if not everywhere constant, a function of u only, so that

$$D_x(I/h_u) : D_x u = D_y(I/h_u) : D_y u = D_z(I/h_u) : D_z u,$$

and if these relations are satisfied, the components of the curl of I vanish, and the polarization is lamellar. Every lamellarly polarized distribution may be divided up into simple polarized shells; if the polarization is not lamellar, but if the directions of this vector and its curl are everywhere perpendicular to each other, the distribution, as we have seen, may be divided up into shells, but these will not be simple.

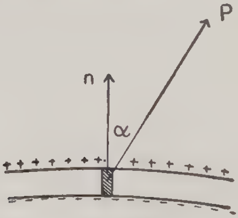

FIG. 54.

The potential function due to a polarized element of moment M has at a point, P, distant r from the element, the value $M \cos \alpha / r^2$, where α is the angle which a line drawn from the element to P makes with the direction of polarization in the element. If a very thin simple shell be divided up into elements of length equal to the thickness (Δn) of the shell and of cross-section equal to an element (ΔS) of one surface of the shell, the moment of each element is $\Delta S \cdot I \Delta n$, and if $I \Delta n$, the constant strength of the shell, be denoted by Φ, the potential function due to the shell has at any point, P, the value limit $\Phi \sum \cos(r, n) \Delta S / r^2 = \Phi\omega$, where ω is the solid angle subtended at P by the boundary of the shell. This value is positive if, in looking out from the vertex P within the conical surface which passes through the boundary of the shell, one sees the positive side of the shell. If, while the strength of the shell is unchanged and the boundary fixed, the shell itself be imagined deformed in any way, the value at P of the potential function due to the shell will be unchanged so long as P is on the same side of the shell. The potential function due to a closed simple shell of any form is zero at every outside point and $\pm 4\pi\Phi$ at every inside point, where the positive sign is to be used if the positive side of the shell is turned inwards.

If P and P' are two points close to each other on opposite sides of a simple, very thin shell, S, of strength Φ, and if V_P and $V_{P'}$ are the values of the potential function at P and P', due to S, we may imagine the shell closed by an additional shell also of strength Φ which shall add to the potential functions at each of the near points P and P' the quantity x. If P is within the closed shell, P' will be outside, so that

$$V + x = 0, \quad V' + x = \pm 4\pi\Phi, \text{ or } V' - V = \pm 4\pi\Phi.$$

The potential function due to an infinitely thin, open or closed, simple polarized shell is, therefore, discontinuous at the shell by $\pm 4\pi$ times the strength of the shell.

The potential energy of a magnetic north pole of strength m at a point, P, near a simple, finite magnetic shell is $\pm m\Phi\omega$, and if P is on the positive side of the shell, $m\Phi\omega$ ergs will be done by the field on the pole if it be carried to infinity by any path. If the pole be carried around the edge of the shell from a point very near the shell on the positive side to a point very near the first but on the negative side, the work done on the pole by the field will be $4\pi m\Phi$ ergs.

In general,

$$m\Phi\omega = \Phi \int\int \frac{m \cos(n, r)}{r^2} dS,$$

where r is the distance *from* dS *to* P, and n is the normal to the shell on the positive side. If the directions of both r and n were considered reversed, the value of the integral would be unchanged, but it would then more clearly represent the surface integral, taken over the shell, of the normal component towards the negative side of the shell of the force due to magnetic pole at P. If, instead of a single pole at P, there is any collection of poles at different points or, indeed, any magnetic distribution, M, the mutual potential energy of the shell and this distribution is equal to Φ times the flux of magnetic force due to M in the *negative* direction through the shell.

A simple magnetic shell in a magnetic field, H_0, due to matter outside the shell tends to move so as to decrease the mutual potential energy of the shell and the field, and this quantity, as we have just seen, is equal to the *negative* of the product of the strength of the shell and the number N of lines (unit tubes) of force due to the field which cross the shell in the *positive* direction. The shell, therefore, tends to move so as to make N as great as possible. If the shell be displaced parallel to itself through a very short distance, du, in any direction, the limit of the ratio of the loss of energy $(+\Phi \cdot dN)$ caused by the displacement to du (*i.e.*, $\Phi \cdot D_u N$) measures the force U, which tends to move the shell in this direction.

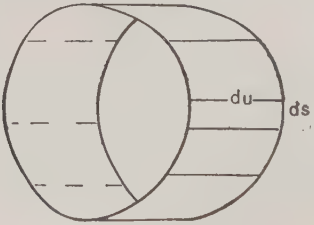

FIG. 55.

If we suppose that the shell in being displaced does not encounter any of the magnetic matter which gives rise to the field, H_0 will be a solenoidal vector within the cylinder generated by the shell, so that the integral of the normal outward component of H_0 taken over the surface of the cylinder will be zero. The shell in its initial and final positions forms the ends of the cylinder, and these together contribute dN to the surface integral, so that the convex surface must contribute $-dN$. If ds is an element, measured in the positive direction about the shell, of the curve which bounds the shell in its original position, and if dS is the element of the convex surface of the cylinder generated by ds,

$$dS = ds \cdot du \cdot \sin(du,\, ds);$$

the magnetic induction through this element due to the magnetic matter outside the shell is

$$H_0 \cdot \cos(n,\, H_0) \cdot \sin(du,\, ds) \cdot du \cdot ds,$$

and this integrated with respect to s is equal to $-dN$, or to $-Udu/\Phi$. Therefore,

$$U = -\Phi \int H_0 \cdot \cos(n,\, H_0) \cdot \sin(du,\, ds) \cdot ds,$$

and the component in any direction (u) of the whole force on the shell may be expressed as a line integral taken around the curve which bounds the shell. The integrand vanishes at any point where u is parallel to H_0 or to ds, but if at any point u happens to be perpendicular to the plane of H_0 and ds, the integrand becomes $H_0 \sin(H_0,\, ds)$, the component of the field perpendicular to ds. If, with this fact in mind, we choose at every point on the curve a direction, p, perpendicular to the plane of H_0 and ds, so that

$$\cos(p,\, ds) = 0 \text{ and } \cos(p,\, H_0) = 0,$$

and remember that $\cos(n,\, ds) = 0$, $\cos(u,\, n) = 0$, we may easily prove that

$$H_0 \cdot \cos(n,\, H_0) \cdot \sin(du,\, ds) = H_0 \cdot \sin(H_0,\, ds) \cdot \cos(p,\, u).$$

This shows that U may be mathematically accounted for by assuming that every element of the *curvilinear boundary* of the shell is urged in a direction perpendicular to the field and to the element, by a force numerically equal to the product of the length of the element, the strength of the shell, and the component perpendicular to the element of the field, H_0.

If the field is due to a single magnetic pole of strength m at a point, P, distant r from ds, the force on the element would be $m\Phi \cdot \sin(r, ds) \cdot ds / r^2$, and the force exerted by the shell on the pole would be accounted for by assuming that every element, ds, of the boundary of the shell contributed an elementary component, $m\Phi \cdot \sin(r, ds)\, ds / r^2$, in a direction perpendicular to the plane of P and ds.

VECTOR POTENTIAL FUNCTIONS OF THE INDUCTION.

Every vector, K, which, except in a given finite region, T, is everywhere continuous, solenoidal, and lamellar, has in simply connected space outside T an easily found scalar potential function, W, which satisfies Laplace's Equation. We may assign to W at pleasure a numerical value at any given point, O, and define the value of W at any other point, O', to be the line integral of the tangential component of K taken along any path from O to O' which does not cut T. The partial derivatives with respect to x, y, and z of W thus defined outside T are evidently equal at every point to the components of K parallel to the coördinate axes, and, since K is solenoidal, $\nabla^2 W = 0$. If K so vanishes at infinity that the limit of the product of its intensity and the square of the distance (r) from any finite point is finite, the limit of $r^2 \cdot D_r W$ is finite, and if we assign to W the value zero at any point at infinity, its value everywhere at infinity will be zero. If K is continuous and if it vanishes at infinity in the manner just described, and is known to be *everywhere* solenoidal and lamellar, it must vanish everywhere; for; if we apply [151] to the harmonic function W within an infinite sphere, it will appear that W, which vanishes at infinity, is identically equal to zero. The vector which represents the force in the case of a charged spherical conductor is solenoidal and lamellar within and without the conductor, and it vanishes properly at infinity, but it is discontinuous at the surface of the sphere. It is usually convenient to assume that the integral of the normal component of a vector, taken over any closed surface at which the vector and its first derivatives are continuous, is equal to the integral of the divergence taken through the space within the surface, even though at some inner surface the vector is discontinuous. On this assumption the vector just mentioned is not solenoidal on the surface of the conductor, for it has there divergence equal in total amount to 4π times the charge.

The line integral of the tangential component of a vector, taken around a closed curve on which this component is continuous, is generally used as a measure of the integral of the normal component of the curl of the vector taken over a cap, S, bounded by the curve, even though at some curve on S the vector ceases to be continuous.

A vector cannot be considered lamellar at a surface where, though its normal component is continuous, some of its tangential components are discontinuous.

If two continuous vectors, U and U', which so vanish at infinity that r^2U and r^2U' have finite limits, have at every point in space equal curls and divergences, and are lamellar and solenoidal outside certain given finite regions, they are identical; for the difference between these vectors is everywhere lamellar and solenoidal, and it vanishes at infinity in such a manner that the product of its intensity and the square of the distance from any finite point is finite. This theorem may be extended to the case where U and U', though not everywhere continuous, have identical discontinuities.

If N_1, ξ_1, η_1, ζ_1 represent the numerical values at the point (x_1, y_1, z_1) of the divergence and the curl components of the vector U, which outside a given region is everywhere continuous, lamellar, and solenoidal, and which so vanishes at infinity that r^2U has a finite limit; if

$$r^2 \equiv (x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2;$$

and if

$$E \equiv -\frac{1}{4\pi}\iiint \frac{\mathrm{N}_1 d\tau_1}{r}, \quad F_x \equiv \frac{1}{4\pi}\iiint \frac{\xi_1 d\tau_1}{r},$$

$$F_y \equiv \frac{1}{4\pi}\iiint \frac{\eta_1 d\tau_1}{r}, \quad F_z \equiv \frac{1}{4\pi}\iiint \frac{\zeta_1 d\tau_1}{r},$$

in which the integrations are to be extended over all space, or at least over all space where U is not lamellar and solenoidal; we know from the theory of the Newtonian potential function, where similar integrals have been studied, that, if N, ξ, η, ζ are the divergence and the curl components of U at (x, y, z),

$$\nabla^2 E = \mathrm{N},\ \nabla^2 F_x = -\xi,\ \nabla^2 F_y = -\eta,\ \nabla^2 F_z = -\zeta.$$

The divergence of the vector F, which has the components F_x, F_y, F_z, is equal to

$$\iiint\limits_{\infty} [\xi_1 \cdot D_x(1/r) + \eta_1 \cdot D_y(1/r) + \zeta_1 \cdot D_z(1/r)]\, d\tau_1,$$

and, since $$D_x(1/r) = -D_{x_1}(1/r),$$

and $$-\xi_1 \cdot D_{x_1}(1/r) = D_{x_1}\xi_1/r - D_{x_1}(\xi_1/r),$$

we may write this by the help of Green's transformation in the form

$$\iiint\limits_{\infty} (D_{x_1}\xi_1 + D_{y_1}\eta_1 + D_{z_1}\zeta_1)/r \cdot d\tau_1$$
$$- \iint [\xi_1 \cdot \cos(x, n) + \eta_1 \cdot \cos(y, n) + \zeta_1 \cdot \cos(z, n)]/r \cdot dS_1,$$

where the second integral is to be taken over the outer boundary of space. The integrand of the triple integral vanishes everywhere, because the vector (ξ, η, ζ), being the curl of another vector, is itself solenoidal. The field of the double integral is in a region where U is lamellar, so that the integral itself vanishes and F is seen to be solenoidal for all values of x, y, and z.

From these results it appears that the vector which has for its components ($D_x E$ plus the x component of the curl of F), ($D_y E$ plus the y component of the curl of F), ($D_z E$ plus the

z component of the curl of F) has everywhere the same curl and the same divergence as U and vanishes like it at infinity, so that it is identically equal to U. D_xE, D_yE, D_zE are the components of a lamellar vector, and the curl of F is solenoidal, so that the vector U, which is not everywhere either solenoidal or lamellar, is everywhere expressible, as was first shown by Helmholtz,* as the sum of a solenoidal and a lamellar vector. The equations

$$U_x = D_xE + D_yF_z - D_zF_y, \qquad U_y = D_yE + D_zF_x - D_xF_z,$$
$$U_z = D_zE + D_xF_y - D_yF_x$$

give any vector, U, which is known to vanish properly at infinity, when its curl components and its divergence are known. If U is solenoidal, E vanishes and F is a vector potential function of U. Every *lamellar vector* has a *scalar potential function* the component of the *gradient* of which, at any point, in any direction, gives the intensity of the component of the vector at that point, in that direction. The component at any point, in any direction, of the *curl* of a *vector potential function* of a *solenoidal vector* gives the intensity of the component of the vector at the given point, in the given direction. Heaviside gives the name "circuital" to a vector which is solenoidal but not lamellar, and the name "divergent" to a vector which is lamellar but not solenoidal.

If ρ_1 is a function of x_1, y_1, z_1, and if r^2 stands for the expression $(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2$, the familiar integral $\iiint \frac{\rho_1}{r}\, dx_1\, dy_1\, dz_1$, extended over all space, is a function of x, y, z, which Prof. J. Willard Gibbs in a remarkable paper † has denoted by the symbol Pot ρ. Using this notation, we may write

$$4\pi E = -\,\text{Pot N},\ \ 4\pi F_x = \text{Pot}\,\xi,\ \ 4\pi F_y = \text{Pot}\,\eta,\ \ 4\pi F_z = \text{Pot}\,\zeta;$$

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* Crelle's *Journal*, Bd. LV, 1858.

† *Elements of Vector Analysis*, § 92. See also Heaviside's *Electrical Papers*, XXIV.

and if we represent by Pot curl U the vector which has for its components Pot ξ, Pot η, Pot ζ, we have the vector equation $4\pi F =$ Pot curl U, and if U is solenoidal, $4\pi U =$ curl Pot curl U. If U is solenoidal, $4\pi U =$ curl curl Pot $U =$ Pot curl curl U, and curl Pot U is a vector potential function of $4\pi U$, or Pot U is a vector potential function of a vector potential function of $4\pi U$. In the case of any polarized distribution whatever, provided there is no intrinsic volume density ρ_0, the induction is solenoidal and has a vector potential function.

II. ELECTROKINEMATICS.

70. Steady Currents of Electricity. When a charged body A is brought up into the neighborhood of a previously uncharged, insulated conductor B, the two kinds of electricity which, according to our provisional theory, exist in equal quantities in every particle of B tend to separate from each other and, as a consequence, free electricity appears on B's surface, some parts of this surface becoming charged positively and other parts negatively. If A is brought into a given position and fixed there, the distribution on the surface of B quickly attains and keeps a value determined by the fact that the whole interior of B must be a region at constant potential, or, in other words, that the resultant force at any point within B due to the free electricities on its surface must be equal and opposite to the force at that point due to all the free electricity outside B. If, now, A with its charge is moved to a new position, the old distribution on B's surface will not in general screen the interior of B from the action of A's charge, and a new separation of electricity within B and a new arrangement or combination of the charge on the surface is necessary before a new state of equilibrium can be established. If A be moved continuously in any manner, there will be a constant attempt on the part of the separated electricities to set up a state of equilibrium, and hence at every point of B there will be, in general, some electrical change going on continually.

If two conductors A and B at different potentials be connected by a fine wire, the whole will form a single conductor, which can only be in a state of equilibrium when the value of the potential function due to all the free electricity in existence is constant throughout its interior, and there will be such a transfer of electricity through the wire as will establish this state of equilibrium in a very short time. If, however, by any device we can furnish unlimited quantities of electricity to A and B in such a way as to keep them at the same potentials as at the beginning, there will be a continual attempt to establish electric equilibrium within the compound conductor consisting of A, B, and the wire, and, as a result, there will be a continual transfer of electricity through the wire.

The transfer of electricity from one place to another through a conductor is a very common phenomenon. Sometimes, as we have seen, electricity traverses the conductor for a short time only; sometimes, however, the transfer goes on indefinitely, and, so far as we can judge from its attendant phenomena, at a constant rate, so that just as much of a given kind of electricity crosses any surface within the conductor in any one second as in any other: such a continuous steady transfer as this is called a "steady current."

The existence of a steady current in a conductor implies a force tending to drive electricity through the conductor; that is, it implies, at least in the absence of moving magnetic masses and of electric currents in the neighborhood of the conductor, free electricity somewhere in existence which gives rise to a potential function not constant throughout the conductor. No part of a conductor through which a steady current is flowing can accumulate free electricity as the time goes on, for such an accumulation increasing with the time would be accompanied by changes which must show themselves outside the conductor. We are led to assume, then, that if any closed surface be drawn inside a conductor which carries a steady current, just as much electricity of a given kind enters the region enclosed by the surface in any interval of time as leaves it during that interval.

We have seen that at every point inside a conductor where there is a resultant electric force there will be an electric separation which will go on as long as the force exists. Experiment seems to show that the rate of separation of quantities of electricity is proportional to the magnitude of the force. Let P be a point of a small plane area ω inside a conductor, and let F be the average value during the interval from t to $t + \Delta t$ of the component of the electric force normal to this area; then in what follows we shall assume that the amount of positive electricity which crosses this surface, in the sense in which the force points, during the interval is $k \cdot \omega \cdot F \cdot \Delta t$, where k is a constant depending only upon the material of which the conductor is composed and upon its physical condition. The average value of this flux per unit of time per unit of surface is, therefore, $k \cdot F$. If, now, ω and Δt are made to grow smaller and smaller in such a manner that P is always a point of ω, F approaches as a limit the negative of the value at P of the derivative, taken in the direction in which F acts, of V, the potential function due to all the free electricity in existence; so that at any instant the value at a point, P, in any direction, n, of the rate of flow of positive electricity across a surface normal to n, per unit of this surface per unit of time, is the value at P of $-k \cdot D_n V$.

It follows from this that if any tube of force be drawn in a conductor which carries a steady current, there is no flow through the sides of the tube. Consider a region shut in by a tube of force and by two equipotential surfaces inside a conductor through which a steady current is flowing. Let ω_1 and ω_2 be the areas of the equipotential ends of the region, and let F_1 and F_2 be the average values of the normal force, taken in the same sense in both cases, over these ends. Applying Gauss's theorem to this region we have $F_2\omega_2 - F_1\omega_1 = 4\pi Q$, where Q is the amount of free electricity, algebraically considered, within the region. If the conductor is homogeneous, the amount of positive electricity which enters — or the amount of negative electricity which leaves — the region by one end per unit of time is $kF_1 \cdot \omega_1$, and the amount which leaves it at the other end is $kF_2 \cdot \omega_2$. These amounts are equal, so that $F_2\omega_2 - F_1\omega_1 = 0$; hence, $Q = 0$, and there is no free electricity at any point within a homogeneous conductor which carries a steady current. The free electricity which gives rise to the potential function the rate of change of which is proportional to the flow of electricity within the conductor, must then lie either outside the conductor, or on its surface, or both. It would not be difficult to prove that there must be a distribution of electricity on parts of the surface of every conductor which carries a steady current and is in contact in some places with an insulating medium; but the fact that a wire through which such a current is passing may be moved about so as to change its position with respect to outside bodies without changing the amount of the current will suffice to make it probable that a part, at least, of the free electricity that we have been considering moves with the wire. Since the density of the free electricity within a conductor which carries a steady current is zero, the potential function V, inside the conductor, must satisfy Laplace's Equation; that is, $\nabla^2 V = 0$. It is easy to see, since there can be no accumulation of free electricity in any conductor which bears a steady current, that the amount of electricity which comes up on one side to the common surface of two such conductors which are in contact must be equal to that which goes away from this surface on the other; that is, at every point of the surface, $k_1 \cdot D_n V_1 = k_2 \cdot D_n V_2$, where k_1 and k_2 are the specific conductivities of the two conductors, and $D_n V_1$ and $D_n V_2$ the values at the point, taken in the same sense in both cases, of the derivatives of V in the direction of the normal to the surface, one on one side of the surface, and the other on the other. It is to be noticed that the boundary between two such conductors may or may not be an equipotential surface. At every point of the common surface of a conductor and an insulating medium $k \cdot D_n V = 0$ or $D_n V = 0$; hence the equipotential surfaces within the conductor cut the surface where the conductor abuts on the insulating medium at right angles.

71. Linear Conductors. Resistance. Law of Tensions. Let us consider the case of a linear conductor, that is, one in which all the lines of force are parallel to each other and to the sides of the conductor, so that every tube of force has a constant cross-section throughout that part of its length which lies in the given conductor. It will appear later on that any right cylindrical conductor, whatever the form of its cross-section, will be a linear conductor, if every point of one of its ends be kept at one constant potential, and every point of the other end at another. It will also be evident that such wires as are ordinarily used for making electrical connections are, to all intents and purposes, except perhaps at the very ends, linear conductors, whether these wires are straight or curved. Let the ends of a homogeneous long uniform straight wire of constant cross-section q, and of length l, be kept respectively at potentials V' and V''. Take the axis of the wire for the axis of x, and the origin at that end of the wire at which the potential function due to all the free electricity in existence is V'; then every line of force inside the wire is parallel to the axis of x; and since there is no force in any direction perpendicular to the axis of x, $D_y V = 0_1$, $D_z V = 0$, and Laplace's Equation, which must be satisfied by V inside the wire becomes $D_x^2 V = 0$, whence $V = Ax + B$; or, since $V = V'$ when $x = 0$, and $V = V''$ when $x = l$,

$$V = \frac{(V'' - V')x}{l} + V'.$$

The steady current c which traverses the wire carries across every right section in the unit of time $-kq \cdot D_x V$ units of positive electricity in the positive direction of the axis of x. That is,

$$c = -kq \cdot D_x V = \frac{kq}{l}(V' - V''),$$

where k is the specific conductivity of the material out of which the wire is made. The quantity l/kq is called the *resistance* of the wire, kq/l its *conductivity*. The quantity k is a function of the temperature. In the case of a pure solid metal at any ordinary temperature a rise of 1° C. will increase $1/k$ by about 0.004 times its own value. This fractional increase is much smaller in the case of some alloys: for "manganin" at room temperatures it is not more than 0.00001.

The analysis of this section assumes that the homogeneous linear conductor is at the same temperature throughout and that it is not surrounded by a changing magnetic field.

It is an important physical principle, first enunciated in a slightly different form by Ohm, that if a fixed portion of the surface of a given homogeneous conductor be kept constantly at potential V_1, and another fixed portion at potential V_2, while the rest of the surface of the conductor is in contact with an insulating medium, the ratio of $V_1 - V_2$ to the steady current which traverses the conductor, — as measured by the quantity of positive electricity per unit of time which either enters the conductor through the surface $V = V_1$ or leaves it through the surface $V = V_2$, — is a quantity independent of V_1 and V_2. This ratio is called the resistance of the conductor under the given circumstances. The resistance of a conductor depends not only upon its shape, the material of which it is composed, and the temperature and other physical conditions of this material, but also upon the shape, size, and position of those portions of the surface which are kept at the potentials V_1 and V_2. The resistance of so much of a tube of force drawn in a conductor which bears a steady current as lies between the equipotential surfaces $V = V_1$ and $V = V_2$ is the ratio of $V_1 - V_2$ to the amount of positive electricity per unit of time which enters the portion of the tube which we have been considering through the surface $V = V_1$, or leaves it through the surface $V = V_2$, or crosses any section of the tube in the direction indicated. Any electric change which, under the same conditions of temperature and pressure, will leave this tube of force still a tube of force and its equipotential ends still equipotential, however the value of the potential function may be changed, will, according to this law of Ohm, leave the resistance the same. Other things being equal, the resistance of a tube of force increases with the length of the tube and diminishes as the section of the tube is made greater.

Suppose that we have a series of linear conductors joined end to end in a closed ring, so that the end of the nth conductor is in contact with the beginning of the first. Let V_m' and V_m'' be the values of the potential function at the beginning and end of the mth conductor, and r_m the resistance of this conductor. Since the same current c must traverse every conductor of the series, we have

$$V_1' - V_1'' = cr_1,\ V_2' - V_2'' = cr_2,\ V_3' - V_3'' = cr_3,\ \cdots V_n' - V_n'' = cr_n;$$

and, if we add them together, we shall get

$$c = \frac{(V_2' - V_1'') + (V_3' - V_2'') + (V_4' - V_3'') + \cdots + (V_1' - V_n'')}{r_1 + r_2 + r_3 + \cdots + r_n},$$

where $V_2' - V_1''$ is the difference between the values of the potential function on opposite sides of the surface common to the second and first conductors, $V_3' - V_2''$ the corresponding difference for the third and second conductors, and so on around the ring. If the sum of these differences is not zero, the circuit is said to be the seat of an electromotive force.

We may here assume that when any two conductors, at the same temperature throughout, but made of different materials, are placed in contact with each other, a discontinuity* of the potential function suddenly appears at their common surface. The amount of this discontinuity, which remains constant after it has once been established, is the same for all points of the common boundary of the two conductors, and is independent of their size and shape, of the extent of surface in contact, and of the absolute values of the potential function on either side of the boundary. We shall represent the sudden fall in the value of the potential function encountered by passing from a conductor made of material A to a conductor made of material B across any point of their common surface by the symbol $A \mid B$. A certain class of substances, to which all metals belong, has the property that if L, M, and N are any three of these substances, all at the same temperature,

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* Although the language of the old "Two Fluid Theory" is used in this chapter, the reader is strongly urged to make himself acquainted with the physical theories now commonly used in accounting for electrical phenomena. See Dr. O. J. Lodge's papers "On the Seat of the Electromotive Force in the Voltaic Cell," printed in the *Philosophical Magazine* for March, April, May, and October of 1885, and his "Modern Views of Electricity," a series of contributions to *Nature*, begun in 1886.

$$L \mid M + M \mid N = L \mid N.$$

This class is said to obey "Volta's Law of Tensions." If a number of conductors made of different kinds of metals all at the same temperature be placed in line, the first in contact with the second, the second with the third, and so on, the algebraic sum of the jumps of the potential function encountered in going from the first conductor to the last through all the others is exactly the same in amount as the single jump which would occur at the common surface of the first and last conductors if they were put directly in contact with each other. Some other substances besides metals obey the Law of Tensions, but most liquids and solutions, whether in contact with each other or with metals, do not obey this law.

The sum of the jumps in the potential function encountered in passing from copper to zinc by way of an iron conductor is the same, if the whole be at one temperature, as the jump encountered in passing directly from copper to zinc. But this is not equal to the sum of the jumps met with in passing from copper to zinc through sulphuric acid.

$$\mathrm{Cu} \mid \mathrm{Fe} + \mathrm{Fe} \mid \mathrm{Zn} = \mathrm{Cu} \mid \mathrm{Zn},$$

but

$$\mathrm{Cu} \mid (\mathrm{H_2SO_4}) + (\mathrm{H_2SO_4}) \mid \mathrm{Zn} \neq \mathrm{Cu} \mid \mathrm{Zn}.$$

The numerator of the expression just found for the intensity of the current which traverses a closed chain of linear conductors is evidently the algebraic sum of the "jumps" in the potential function encountered by travelling in the direction in which the current is supposed to move, from the first conductor to the last through all the others, and reckoning the jump at any boundary positive if the value of the potential function is increased as one crosses the boundary. If all the conductors which form the circuit are metallic and all at the same temperature, whether or not they are all made of the same kind of metal, this numerator is zero, and it follows that in order that a steady current may traverse a circuit of conductors, one at least of the conductors must disobey the Law of Tensions.

The same formulas apply to a circuit composed of conductors of any form if each of the common surfaces of contiguous conductors is equipotential.

Every slender tube of force in a homogeneous conductor which carries a steady current is also a tube of flow and constitutes a *current filament.* We shall hereafter apply the term *linear* only to conductors which have very small cross-sections.

72. Electromotive Force. We have seen that if a number of homogeneous conductors made of different materials be connected in series to form a heterogeneous conductor K, there will be discontinuities in the electrostatic potential function within K at the common surfaces of adjacent conductors. If an equipotential surface A near one end of K be kept at potential V_A, and an equipotential surface B near the other end of K, at potential V_B, and if the algebraic sum of the discontinuities of potential between A and B, counting a step up as positive, is E, the current in K from A to B will be $(V_A - V_B + E)/r$, where r is the resistance between A and B. In such a case as this, $V_A - V_B$ is called the *electrostatic* or *external electromotive force*, and E the *internal* or *intrinsic electromotive force.* If K forms a closed circuit, all the electromotive force may be regarded as internal. In this connection it should be said that, although physicists are not all in agreement as to the magnitude of the discontinuity of potential at the surface of contact of any two given dissimilar conductors, there is no difference of opinion as to the algebraic sum of these discontinuities in the case of any closed circuit.

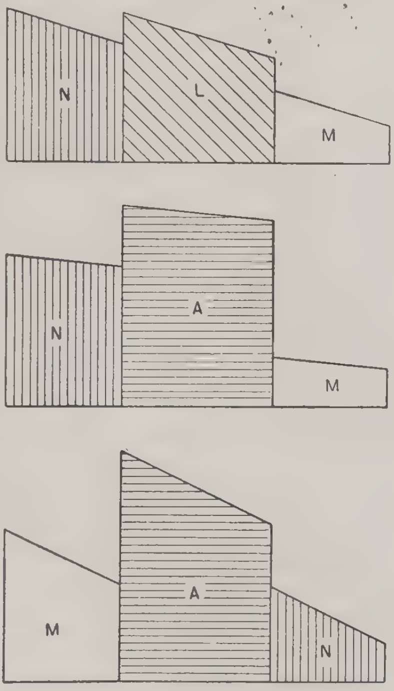

FIG. 56.

If one end of a heterogeneous cylindrical conductor K, of given resistance r, formed of homogeneous cylindrical conductors in series, be kept at a given potential V_1 and the other end at the given potential V_2, the value of the potential function will depend very much upon the constitution of K. Three different cases are illustrated in Fig. 56, in which abscissas represent resistances and ordinates the corresponding values of V. In these figures A is supposed to be an electrolyte, while L, M, N are metals: $V_1 = 2$, $V_2 = 0.5$, $A \mid N = 0.8$, $A \mid M = 1.8$, $N \mid M = 0$. The current strength (indicated by the slope of the line which gives the value of V) is evidently different in the different diagrams.

Fig. 57 represents V in a long chain made of two metals P, Q, and an electrolyte R, such that $R \mid P$ is small, $R \mid Q$ still smaller, and $P \mid Q$ zero. Here the ends are at the same potential, and there are no great potential differences anywhere in the chain, but the current (as indicated by the slope of the V line) is large, as is the sum of the small discontinuities which go to make up the electromotive force in the chain.

A galvanic battery may be regarded as a chain of three or more generally non-linear conductors, at least one of which disobeys the Law of Tensions. The algebraic sum of the jumps in the potential function encountered by starting at that pole of a galvanic battery at which the potential is less, and passing to the other pole through the battery, is the electromotive force of the battery. The difference of potential between copper wires attached to the open poles of the battery, measures this electromotive force. Chemical action goes on inside every battery when its poles are closed; some of its solutions are decomposed, and the products of this decomposition often appear at the boundaries of the liquid conductors inside the battery and decrease the electromotive force by changing the amount of jump in the potential function at each of these boundaries. For this reason the electromotive force of a battery in action may be much less than when the poles are open.

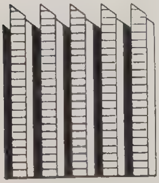
FIG. 57.

If two points, P and Q, in a network of conductors which carry a steady current, be connected by an additional wire conductor, K, containing a battery of such electromotive force, e, and so directed as to prevent any current from passing through K, e measures the difference of potential between P and Q. It is easy to show that when the poles of a battery are closed by a conductor of resistance R, the difference between the values of the potential function at the ends of this conductor is $RE / (B + R)$, where E is the electromotive force of the battery under the given circumstances, and B the resistance of the conductors which make up the battery itself. The steady current which flows through the circuit carries $E/(B+R)$ units of positive electricity across every cross-section per unit of time. With a given battery the intensity of the current can be changed very much, of course, by increasing or decreasing the resistance of that part of the circuit which lies outside the battery.

In the centimetre-gramme-second system of electrostatic [E.S.] absolute units, the unit of electric quantity is that quantity of electricity which, if it could be concentrated at a point in air, would repel a like quantity concentrated at a point 1 centimetre from the first with a force of 1 dyne. This unit is found inconveniently small, however, when one has to deal with such steady currents as are usually met with in practice, and the *coulomb*, which is equal to about 3×10^9 of these absolute units, is the practical unit of quantity most frequently used.

The absolute E.S. unit of current carries the absolute unit of electricity past any point in its course each second. A current of a coulomb per second (equivalent to 3×10^9 of these absolute current units) is called an *ampere.*

The absolute E.S. unit of resistance is 9×10^{11} times as large as the practical unit called the *ohm.* The latter is the resistance of a column of pure mercury 1 square millimetre in section and 106.3 centimetres long, at 0° C. The resistance at 0° C. of a wire of pure copper 1 millimetre in diameter and 1 metre long is about 0.01642 ohm.

The absolute E.S. unit of difference of potential is equivalent to 300 practical units. The practical unit, called the *volt,* is such that if the two ends of a wire of 1 ohm resistance were kept at 1 volt difference of potential, the steady current which traversed the wire would carry past any cross-section 1 coulomb of electricity per second.

A condenser which requires 1 coulomb of electricity to charge it, so that the difference of potential between its poles is 1 volt, is said to have a capacity of 1 *farad.* A convenient unit of capacity is the *microfarad* or the millionth of a farad. It is equivalent to 900,000 absolute E.S. units of capacity. The capacity of a conducting sphere 9 kilometres in radius would be 1 microfarad, that of the earth something over 700 microfarads. The capacity of a nautical mile of such ocean telegraph cable as is usually laid may be taken to be about $\frac{1}{3}$ microfarad.

73. Kirchhoff's Laws. The Law of Divided Circuits. From what has been proved in the preceding sections about conductors which carry steady currents, follow two theorems of much practical importance, called Kirchhoff's Laws.

I. If several wires which form part of a network of conductors carrying a steady current meet at a point, the sum of the intensities of all the currents which flow towards the point through these wires is equal to the sum of all those which recede from it; or, in other words, the algebraic sum of all the currents which approach the point through the wires which meet there is zero.

II. If, out of any network of wires which form a complex conductor and carry a steady current, a number of wires which form a closed figure be chosen, and if, starting at any point, we follow the figure around in either direction, calling all currents which move with us positive, and all discontinuities of the potential function which lift us from places of lower potential to places of higher potential positive, the algebraic sum of the products formed by multiplying the resistance of each conductor by the current running through it, is equal to the algebraic sum of the jumps in the potential function which we encounter in going completely around the figure.

The first of these laws is an immediate consequence of the fact that there can be no growing accumulation of free electricity anywhere in a circuit which bears a steady current. To prove the second law, let $a_1, a_2, a_3, \cdots a_n$ be n linear conductors, which, taken in order, form a closed figure, itself a part of a complex conductor which carries a steady current. In passing from a_1 to a_n through all the other conductors, let

V_j' and V_j'' be the values of the potential function at the beginning and end of the jth conductor, and let r_j and c_j be respectively the resistance of this conductor and the value of the current running through it. Then, from the definition of the term "resistance," we have the following equations:

$$V_1' - V_1'' = c_1 r_1 ; \quad V_2' - V_2'' = c_2 r_2 ;$$
$$V_3' - V_3'' = c_3 r_3 ; \quad \cdots V_n' - V_n'' = c_n r_n ;$$

or, adding them all together,

$$\begin{aligned} &c_1 r_1 + c_2 r_2 + c_3 r_3 + \cdots + c_n r_n \\ &\quad = V_2' - V_1'' + V_3' - V_2'' + V_4' - V_3'' + \cdots + V_1 - V_n'', \end{aligned}$$

which is the statement of this law.

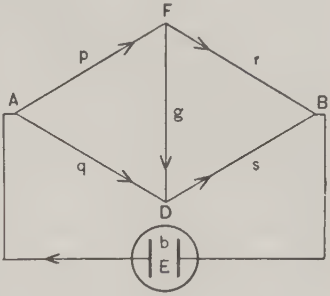

FIG. 58.

If electricity is free to pass from a point P to another point P' by two wires of resistance r_1 and r_2, respectively, and if a steady current be flowing from P to P', the current will be divided between the two wires in the inverse ratio of their resistances or in the direct ratio of their conductivities. For, if V and V' be the values of the potential function at P and P', we have $V - V' = c_1 r_1$ and $V - V' = c_2 r_2$, whence $c_1 : c_2 = r_2 : r_1$.

Moreover, $$c_1 + c_2 = (V - V')\left(\frac{1}{r_1} + \frac{1}{r_2}\right);$$

or, $$\frac{V - V'}{c_1 + c_2} = \frac{1}{\dfrac{1}{r_1} + \dfrac{1}{r_2}}.$$

The expression in the second number of the last equation is, by the definition of the term, the resistance of the compound conductor formed of the two which join P and P'. It is evident that ~~the con~~ductivity of this conductor is the sum of the conductivities of the two wires of which it is composed.

If n conductors be joined up in parallel to form a compound conductor, the conductivity of the latter is the sum of the conductivities of the constituents, and its resistance is the reciprocal of the sum of the reciprocals of their resistance.

If four conductors the resistances of which are p, q, r, and s form a quadrilateral (Fig. 58) one pair of vertices of which are connected by a wire of resistance g and the other pair by a conductor of resistance b containing a battery of electromotive force E, we have an arrangement of much practical importance, which is often called Wheatstone's Net. If we denote the strength of the current through the cell, in the direction indicated by the arrow in the figure, by C, and the currents in the other conductors by C_p, C_q, C_r, C_s, and C_g respectively, Kirchhoff's Laws yield the equations

$$\begin{aligned}
&C = C_p + C_q = C_r + C_s,\ \ C_p = C_g + C_r,\\
&C_q = C_s - C_g,\ \ p \cdot C_p - q \cdot C_q + g \cdot C_g = 0,\\
&g \cdot C_g - r \cdot C_r + s \cdot C_s = 0,\ \ b \cdot C + q \cdot C_q + s \cdot C_s = E.
\end{aligned}$$

If we substitute the values of C, C_p, C_q obtained from the first three equations in the last three, we shall get a system of three linear equations involving the three unknown quantities C_g, C_r, C_s, which can be easily solved. These equations are

$$\begin{aligned}
&(p + g + q)\, C_g + p \cdot C_r - q \cdot C_s = 0,\\
&g \cdot C_g - r \cdot C_r + s \cdot C_s = 0,\\
&- q \cdot C_g + b \cdot C_r + (b + q + s)\, C_s = E,
\end{aligned}$$

and if we denote the determinant of the coefficients,

$$\begin{aligned}
&-\{ pr(q + s) + qs\,(p + r) + b\,(p + q)\,(r + s)\\
&\qquad + g\,[b\,(p + q + r + s) + (q + s)\,(p + r)]\}
\end{aligned}$$

or

$$\begin{aligned}
&-\{ gr\,(b + q + s) + qb\,(g + r + s) - ps^2 + qrs\\
&\qquad + gbs + p\,(g + r + s)\,(b + q + s)\},\ \text{by } -\Delta,
\end{aligned}$$

it is easy to see that

$$\begin{aligned} C_g &= E\,(qr - ps)\,/\,\Delta, \\ C_r &= E\,(gq + sp + sg + sq)\,/\,\Delta, \\ C_s &= E\,(gp + rp + rg + rq)\,/\,\Delta, \\ C_p &= E\,(qr + gq + sg + sq)\,/\,\Delta, \\ C_q &= E\,(gr + gp + rp + ps)\,/\,\Delta. \\ C\ &= E\,(gq + sp + sg + sq + pg + rp + rg + rq)\,/\,\Delta. \end{aligned}$$

The resistance (R) of the net $pqrsg$, computed from the equation $C = E\,/\,(b + R)$, is

$$\frac{[g\,(q+s)\,(p+r) + pr\,(q+s) + qs\,(p+r)]}{[g\,(p+q+r+s) + (p+q)\,(r+s)]}.$$

If no current passes through the resistance g, we have $qr = ps$, $C_p = C_r$, $C_q = C_s$, and, as we may see by multiplying out and cancelling,

$$C_r\,/\,C_s = (q+s)\,/\,(p+r),\ \ C_r\,/\,C = (q+s)\,/\,(p+q+r+s),$$
$$\text{and } C_s\,/\,C = (p+r)\,/\,(p+q+r+s).$$

It is evident, from an inspection of the Kirchhoff equations belonging to the three cases, that if the resistances of the linear conductors which go to make up a given network are fixed, and if $C_1, C_2, C_3, \cdots$ are the currents in the different members when these members contain the electromotive forces $E_1, E_2, E_3, \cdots$ and $C_1', C_2', C_3', \cdots$, the corresponding currents when the electromotive forces are $E_1', E_2', E_3', \cdots$, $C_1 + C_1'$, $C_2 + C_2'$, $C_3 + C_3', \cdots$ would be the currents if the electromotive forces were $E_1 + E_1', E_2 + E_2', E_3 + E_3', \cdots$.

Let P and Q, any two points in a network of linear conductors some or all of which contain electromotive forces, be at potentials V_P, V_Q respectively, and let the resistance of the whole network when the current enters at one of these points and goes out at the other be r_0, then if P and Q be connected by an additional wire W of resistance r, the current in this wire will be $(V_P - V_Q)\,/\,(r_0 + r)$ in the direction from P to Q. For if (1) W contained an electromotive force $(V_P - V_Q)$ directed from Q to P, the rest of the network being unchanged, no current would pass through W, and the other currents would not be altered by the introduction of W; and if (2) W contained the electromotive force $(V_P - V_Q)$ directed from P to Q, and if all the other electromotive forces in the original network were annihilated, leaving the resistances unchanged, a current $(V_P - V_Q) / (r_0 + r)$ would flow through W from P to Q: the given arrangement can be regarded as formed by superposing case (1) upon case (2).

74. The Heat developed in a Circuit which carries a Steady Current. Given, in a region not exposed to magnetic changes, a chain of n conductors, each in itself homogeneous, and at a uniform temperature throughout; let a portion A of the surface of the first be kept, by means of some external agency, at potential V_A, and a portion B of the surface of the last at a lower potential V_B, while the rest of the outer surface of the chain abuts upon non-conducting media. $S_{k,\,k+1}$, the surface of separation between the kth and the $(k+1)$th conductors, may or may not be equipotential, but if these conductors are of different materials, we must expect to find at all points of this surface a uniform discontinuity, $E_{k,\,k+1}$, of potential. In following down from A to B an infinitesimal tube of flow which carries the steady current ΔC, we start at potential V_A, leave the first conductor at potential V_1'', enter the second conductor at potential V_2', leave it at V_2'', enter the third conductor at V_3', and so on. Every second in the kth conductor, ΔC absolute units of electricity are lowered from potential V_k' to potential V_k'' and $\Delta C(V_k' - V_k'')$ units of work (representing loss of electrostatic energy) are done by the electrostatic field upon the electricity which moves with the current: this energy appears as heat in this conductor. The work thus done in the whole chain is

$$\Delta C(V_A - V_1'' + V_2' - V_2'' + V_3' - V_3'' + \cdots + V_n' - V_B),$$

or, since $$V_{k+1}' - V_k'' = E_{k,\,k+1},$$

$$\Delta C(V_A - V_B + E_{1,2} + E_{2,3} + \cdots + E_{n-1,\,n}) = \Delta C(V_A - V_B + E).$$

This energy all appears as heat in the conductors which form the chain.

At the surface $S_{k,\,k+1}$, ΔC units of electricity are raised every second from potential V_k'' to potential V_{k+1}'. The work thus done every second is $\Delta C \cdot E_{k,\,k+1}$, and, by virtue of similar processes at all the surfaces of discontinuity, the electrostatic energy is increased in this way every second by $\Delta C \cdot E$. The net loss in electrostatic energy in the chain per second is, therefore,

$$(V_A - V_B)\,\Delta C,$$

which is otherwise evident. Taking into account all the current filaments which go to form the steady current C, we see that an amount of energy equivalent to $C(V_A - V_B + E)$ appears as heat in the conductors which form the chain, and that an amount of electrostatic energy equal to EC is furnished to the chain. If the chain is closed and if, going around it in the direction of the steady current C, we denote by E the algebraic sum of the discontinuities of potential, counting a step up as positive, we shall find that the energy EC appears as heat in the conductors and that since the circuit is at the same temperature throughout, this is furnished by chemical action in the chain. If r is the total resistance of the chain, $C = E/r$ and $EC = C^2 r$. This result represents ergs or joules, according as E, C, and r are measured in absolute electrostatic units or in volts, amperes, and ohms: a joule is equivalent to 10^7 ergs.

If the chain contains a battery of electromotive force E_0 in the direction of the steady current C, and if there are in the chain outside the battery discontinuities of potential which, reckoned against the current, amount algebraically to E',

$$E = E_0 - E', \quad C = (E_0 - E')/r,$$

and the energy used in heating the chain is $(E_0 - E')\,C = C^2 r$: when we wish to regard the battery as the source of this energy, it is convenient to write the last equation in the form $E_0C = C^2r + E'C$, and to say that of the whole energy, E_0C, furnished by the battery, C^2r, which appears as heat in the conductors which form the circuit, is used in maintaining the current, and $E'C$, in overcoming the counter-electromotive force E'. If a cell of electromotive force E_0 be joined up with a number of metallic conductors all at the same temperature to form a simple circuit of total resistance r, the current will be $C_0 = E_0/r$, and the whole energy, $E_0C_0 = C_0^2r$, furnished each second by the battery, will appear as heat in the circuit. If, however, while the total resistance of the circuit remains unchanged, the battery be called on to do each second an amount W of outside work of any kind (such, for instance, as that involved in decomposing an electrolyte in the external circuit), the steady current will have a value C smaller than C_0, the whole energy E_0C furnished each second by the cell will be a fraction of E_0C_0, and the portion of it C^2r, which appears as heat in the circuit, a smaller fraction of C_0^2r. The difference between E_0C and C^2r will be equal to W, and this equation determines C.

If a given steady current C is to be conveyed partly by a conductor of resistance r_1 and partly by a parallel conductor of resistance r_2, and if the portions carried by these conductors are C_1 and C_2 respectively, the amount of heat developed per second in the conductors will be $u = C_1^2r_1 + C_2^2r_2$. If C_1, and consequently C_2, be changed so as to keep their sum equal to the constant C, u will, in general, change, and we shall have

$$D_{C_1}u = 2\,C_1r_1 + 2\,C_2r_2 \cdot D_{C_1}C_2 = 2\,(C_1r_1 - C_2r_2):$$

u, which is sometimes called the *dissipation function*, will, therefore, be a minimum if the current is divided between r_1 and r_2 as it would be if the conductors were connected at the ends. It is easy to prove that if a given steady current be led into a given network of metallic conductors, at a uniform temperature, from without, the distribution of this current in the network will be such as to make the dissipation function as small as possible. If, for instance, a steady current C be led into the network represented by $ABDF$ in Fig. 58 at the point A and out again at B, we have

$$C_r = C - C_s,\ \ C_p = C_g + C - C_s,\ \ C_q = C_s - C_g,$$

and u is equal to

$$p(C_g + C - C_s)^2 + q(C_s - C_g)^2 + r(C - C_s)^2 + s \cdot C_s^2 + g \cdot C_g^2.$$

If we equate to zero the partial derivatives of u with respect to C_s and C_g, we shall get two necessary conditions for a minimum : the equations thus obtained are

$$\begin{aligned} (p + g + q)\, C_g - (p + q)\, C_s &= -pC, \\ -(p + q)\, C_g + (p + q + r + s)\, C_s &= (p + r)\, C, \end{aligned}$$

whence

$$\begin{aligned} C_g / C &= (qr - ps) / (gq + sp + sg + sq + pg + rp + rg + rq), \\ C\ / C &= (gp + rp + rg + rq) / (gq + sp + sg + sq + pg + rp + rg + rq), \end{aligned}$$

etc., which are equivalent to equations already found.

If the conductors $r_1, r_2, r_3, \cdots r_n$ which form any network, complete or not, and carry currents $C_1, C_2, C_3, \cdots C_n$, contain electromotive forces $E_1, E_2, E_3, \cdots E_n$ which have the directions assumed for the currents, the currents are such as to make, not the dissipation function, but

$$W \equiv u - 2(C_1E_1 + C_2E_2 + C_3E_3 \cdots C_nE_n)$$

a minimum. In the case of the complete Wheatstone's Net,

$$\begin{aligned} W \equiv b(C_r + C_s)^2 + p(C_g + C_r)^2 + q(C_s - C_g)^2 + g \cdot C_g^2 \\ + r \cdot C_r^2 + s \cdot C_s^2 - (C_r + C_s)\, E, \end{aligned}$$

and the equations formed by equating to zero the partial derivatives of W with respect to C_g, C_s, and C_r yield the values for the currents given in the last section.

75. Properties of the Potential Function inside Conductors which carry Steady Currents. If at any time t, positive electricity is passing through a linear conductor in one direction at the rate P, and negative electricity in the other direction at the rate N, the current strength is $P + N$ in the first direction. Since there is no free electricity inside a homogeneous conductor which carries what we have called a steady current, it is customary to assume, when one uses the language of the "Two Fluid Theory," that such a current consists of a flow of positive electricity in one direction at every point, and an equal flow of negative electricity in the opposite direction. We shall avoid much circumlocution, however, and we shall introduce no error into our numerical computations if we speak as if the whole current were due to the motion of positive electricity. If the value of the potential function within a conductor which bears a steady current is given, all the circumstances of the flow in the conductor are fixed. Positive electricity flows into the conductor from without through all parts of the surface where the derivative of the potential function, taken in the direction of the exterior normal, is positive, and out of it through all parts of the surface where this derivative is negative. At all points where the conductor abuts on an insulating medium, the derivative is zero: it may be zero at other points also. There can be no closed equipotential surface lying wholly inside a conductor which carries a steady current, unless there is some constant source of positive or of negative electricity within this surface, for the whole flow of electricity algebraically considered, per unit of time, through such a surface from within outwards, is equal to k times the surface integral of the intensity of the component of force in the direction of the exterior normal, and this is not zero. There must then be such a constant source of free electricity within the surface as shall furnish just as much per unit of time as the current carries away.

Although it is not very easy to prove analytically that — given a homogeneous conductor and certain portions A, B of its surface which are to be kept at potentials V_A, V_B, while at all other portions the value of the derivative of the potential function taken in the direction of the exterior normal is to be zero — there exists a function which (1) satisfies these surface conditions, and which (2) inside the conductor satisfies Laplace's Equation, and with its first space derivatives is continuous and single-valued, it is nevertheless clear from physical considerations that one such function exists, namely, the potential function inside the conductor when A, B are kept at the given potentials and the rest of the surface is exposed to an insulating medium. For practical purposes we need to prove that this is the only function which satisfies the given conditions. Suppose for the sake of argument that two such functions, V and W, exist, and call their difference u. The function u, then, satisfies condition (2) and is itself equal to zero, or else has its derivative in the direction of the exterior normal equal to zero at every point of the surface. Applying Green's Theorem in the form of Equation 151 to u, we find that the quantity $(D_x u)^2 + (D_y u)^2 + (D_z u)^2$, which can never be negative, must be zero at every point within the conductor, so that $D_x u$, $D_y u$, and $D_z u$ must vanish and u be a constant throughout the space within the surface. Now at portions of the surface itself, u is zero, hence it must be equal to zero everywhere inside the conductor, and $V = W$. If by any means, then, we find a function which satisfies the surface conditions and the general space conditions characteristic of the potential function inside a certain conductor carrying a steady current under given surface conditions, this function is itself the potential function.

Any surface supposed drawn in a conductor which carries a steady current in such a way that the derivative of the potential function taken normal to this surface is zero shall be called a *surface of flow.*

If a conductor which under given surface conditions carries a steady current be cut in two by means of a surface of flow, and if the two parts be separated while the surface conditions on what was the bounding surface of the old conductor remain the same as before, and the fresh surfaces now abut on an insulating medium, the state of flow at every point inside each part of the conductor will be just the same as before, for the values of V and $D_n V$ on the surface of the new conductors are what they were before separation, and V must have its old values at all inside points.

When a conductor is cut in two by a surface of flow the fresh surfaces exposed receive a statical charge of free electricity, and the charges on what was the bounding surface of the original conductor are in part changed so that it is only *within* the parts of the old conductor that the effect of the separation is nil after the currents have become again steady.

If two mutually exclusive closed surfaces S_1 and S_2, kept, respectively, at uniform potentials V_1 and V_2, are the electrodes of an infinite homogeneous conductor K, of specific conductivity k, which fills all space outside these surfaces and is at potential zero at infinity; if, moreover, the steady flow outward through S_1 or inward through S_2 is equal to C, the current vector in K is everywhere equal to what the electrostatic force would be if K were air and if S_1 and S_2 had charges $C/4\pi k$ and $-C/4\pi k$ so distributed as to bring them to potentials V_1 and V_2 respectively.

In most of the preceding discussion we have tacitly assumed the separate conductors considered to be homogeneous, and we shall continue to do so in the following sections unless the contrary is stated. We have to consider briefly, however, in the remainder of this section isotropic conductors which have in different parts different specific resistances.

If the specific conductivity k of an isotropic conductor which carries a steady current can be represented by a positive scalar point function, and if the components, parallel to the coördinate axes, of the vector q which represents the current strength, are u, v, and w, we may state the fact that there is no growing accumulation of free electricity in any portion of the conductor bounded by the surface S by the equation

$$\int\int q \cos(q, n)\, dS \equiv \int\int q\left[\cos(x, n)\cdot\cos(x, q)\right.$$

$$\left.+ \cos(y, n)\cdot\cos(y, q) + \cos(z, n)\cdot\cos(z, q)\right] dS$$

$$\equiv \int\int [u \cos(x, n) + v \cos(y, n) + w \cos(z, n)]\, dS$$

$$\equiv \int\int\int [D_x u + D_y v + D_z w]\, dx\, dy\, dz = 0.$$

Here the double integrals are to be extended over the whole of S, and the triple integrals over all the space included by S. Since S is arbitrary, the integrand of the triple integrals must be equal to zero at every point within the conductor, so that

$$D_x u + D_y v + D_z w = 0 \qquad [198]$$

and q is a solenoidal vector.

At every point within the conductor,

$$u = -kD_x V,\ v = -kD_y V,\ w = -kD_z V,$$

so that

$$D_x(k\cdot D_x V) + D_y(k\cdot D_y V) + D_z(k\cdot D_z V) = 0, \quad [199]$$

or $k\cdot\nabla^2 V + (D_x k\cdot D_x V + D_y k\cdot D_y V + D_z k\cdot D_z V) = 0.$ [200]

If k is constant, V satisfies Laplace's Equation, and in this special case, as we already know, none of the free electricity which gives rise to the potential function V is within the conductor.

Given an analytic, scalar, positive point function k and a closed analytic surface S, it is easy to prove by the help of [149] that there cannot be two different functions, V_1 and V_2, which (1) with their first derivatives are continuous within S and at every point in this region satisfy the equation

$$D_x(k\cdot D_x V) + D_y(k\cdot D_y V) + D_z(k\cdot D_z V) = 0,$$

(2) on the given portions S_1 and S_2 of S have at each point equal values, and (3) on the rest of S have at every point equal normal derivatives.

The differential equations of the current lines are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}.$$

At a surface of separation between two conductors which carry a steady current the normal components of the current and the tangential components of the electrostatic force are continuous. If θ_1 and θ_2 are the angles which the resultant electrostatic forces F_1 and F_2 make with the normal on the two sides of such a surface at any point,

$$k_1F_1 \cos\theta_1 = k_2F_2 \cos\theta_2 \text{ and } F_1 \sin\theta_1 = F_2 \sin\theta_2,$$

whence, by dividing the members of the first of these equations by the corresponding members of the second, $\dfrac{\tan\theta_1}{k_1} = \dfrac{\tan\theta_2}{k_2}$, an equation which shows how the current lines are refracted at the surface. At a surface of separation between copper and manganin where the ratio of the conductivities is about 30,

$$\theta_1 = 27^\circ\, 42' \text{ when } \theta_2 = 1^\circ, \text{ and } \theta_1 = 69^\circ\, 09' \text{ when } \theta_2 = 5^\circ.$$

If n_1 and n_2 represent normals drawn from any point of the surface of separation between two conductors which are carrying a steady current into the first and the second conductor respectively,

$$k_1 D_{n_1} V_1 + k_2 D_{n_2} V = 0. \qquad [201]$$

76. Method of finding Cases of Electrokinematic Equilibrium. If w is a single-valued, generally continuous solution of Laplace's Equation, $Aw + B$, where A and B are constants, is another such function which has the same level surfaces as w. If an area be chosen on one of these surfaces, it is possible to draw through every point of its perimeter a line, defined by the equations $dx / D_x w = dy / D_y w = dz / D_z w$, which shall cut orthogonally all the level surfaces of w which it meets. All these lines form a tubular surface such that the normal derivative of w at every point of it is zero. If T is a portion of space bounded by such a tube and by portions, S', S'', of two of w's level surfaces on which it has the values w' and w'' respectively, w is identical with the potential function that would govern the flow within any homogeneous conductor of the form T if the surface S' and S'' were kept at potential w'

and w'', while the rest of the boundary was a surface of flow. Moreover, $Aw + B$, where A and B can be chosen at pleasure, must be the potential function within a homogeneous conductor of the form T, if the surface S' and S'' were kept at potentials $Aw' + B$, $Aw'' + B$ respectively, the rest of the boundary being a surface of flow. By using different pairs of level surfaces of w and tubes of different forms, it is possible with the help of this one function to study the laws of steady flow inside conductors of many different shapes and to obtain results some of which may happen to be practically interesting. For instance, $w = \frac{c}{r} + d$, where c and d are constants and r the distance from a fixed origin O to the point (x, y, z), gives the value of the potential function inside a conductor bounded by two spherical surfaces of radii a and b having O as their common centre when these surfaces are kept respectively at potentials $\frac{c}{a} + d$ and $\frac{c}{b} + d$. In this case the whole amount, per unit of time, of positive electricity which enters the conductor through the surface $r = a$ crosses every equipotential spherical surface within the conductor and leaves it by the surface $r = b$ is $4\pi ck$, where k is the specific conductivity of the material out of which the conductor is made. The resistance of the conductor is, by definition,

$$\frac{\frac{c}{a} - \frac{c}{b}}{4\pi ck} = \frac{b - a}{4\pi kab},$$

a quantity independent of c and d.

It is evident that any conical surface the vertex of which is O will be in this case a surface of flow, and that the function $w = \frac{c}{r} + d$ governs the flow in any piece cut out of the spherical shell just considered by such a surface. It is easy to see that if ω is the solid angle of the cone, the resistance of the portion of the conductor cut out will be $\frac{b - a}{k\omega ab}$.

Again, the equation $V = c\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + d$, where r_1 and r_2 are the distances of the point (x, y, z) from the fixed points O_1 and O_2, gives us the potential function inside an infinite conductor bounded in part by the surfaces $\frac{1}{r_1} - \frac{1}{r_2} = a$ and $\frac{1}{r_1} - \frac{1}{r_2} = b$, when the first is kept at potential $ac + d$, the second at potential $bc + d$. In this case the surface $V = d$ is a plane bisecting at right angles the straight line $O_1 O_2$. Larger and smaller values of V than this give closed surfaces, each of which surrounds one of the points and leaves the other outside. For very large values of V, if c is positive, the equipotential surfaces are very small, nearly spherical surfaces surrounding O_1.

To find the amount of positive electricity which enters the conductor under consideration, per unit of time, through the surface $V = ac + d$, where ac shall be positive, we must integrate over this surface $-kD_nV$ or $-kc\left[D_n\left(\frac{1}{r_1}\right) - D_n\frac{1}{r_2}\right]$. According to Green's Theorem, the resulting integral is exactly the same as that taken over any other closed surface, large or small, which surrounds O_1 and leaves O_2 outside. Let us consider, then, a spherical surface of radius $\epsilon < O_1O_2$ whose centre is at O_1. The required integral in this case is $-4\pi\epsilon^2 k$ times the average value of $D_{r_1}V$ taken over the spherical surface; or, since r_1 for all points on this surface is equal to ϵ,

$$4\pi\epsilon^2 kc\left[\frac{1}{\epsilon^2} - \text{average value of } D_{r_1}\left(\frac{1}{r_2}\right)\right].$$

If, now, ϵ be made smaller and smaller, $D_{r_1}\left(\frac{1}{r_2}\right)$ always has some finite value for every point on the surface of the sphere surrounding O_1, and the expression just given approaches the limiting form $4\pi kc$. Hence, $4\pi kc$ units of positive electricity enter the given conductor through the surface $V = ac + d$ in every second, whether this surface is large or small. The resistance of the conductor between the surfaces $V = ac + d$ and $V = bc + d$ is, by definition of the term, $\frac{a - b}{4\pi k}$.

If a and b are made very large and equal, with opposite signs, the two surfaces through which electricity enters and leaves the conductor become very nearly coincident with spherical surfaces of radius $\epsilon = \frac{1}{a}$ drawn about O_1 and O_2 respectively. The resistance of the conductor in this case is $\frac{1}{2\pi k\epsilon}$. Considerations of symmetry show that any plane which contains the line O_1O_2 is a surface of flow. If we cut the conductor in two by such a plane, we shall have an infinite conductor with two nearly hemispherical electrodes sunk in its plane surface. The resistance of this part of the whole conductor is $\frac{1}{\pi k\epsilon}$, a quantity independent of the distance apart of the electrodes. This is nearly the case of two poles of a battery sunk in the earth.

Again, the expression

$$V = c \log \frac{r_1}{r_2} + d,$$

where r_1 and r_2 are the distances of a point P in space from any two parallel straight lines, A and B, is a solution of Laplace's Equation which, with its derivatives, vanishes at an infinite distance from these lines and which is constant all over any one of a double system of circular cylindrical surfaces (Fig. 59), some of which surround one of the given lines and some the other. This function, then, when c and d are properly determined, is the potential function within an infinite lamina, either thick or thin, when that lamina is perforated perpendicularly to its plane by two circular cylindrical holes, the curved surfaces of which are kept at given constant potentials. $2\pi kc$ units of positive electricity per unit of time per unit of thickness of the lamina enter the conductor through one of the cylindrical surfaces, and the same amount leaves it by the other surface. The resistance of the lamina is then the difference between the values of the potential function at the electrodes divided by $2\pi kc$ times the thickness of the lamina.

These examples will serve to show how we may discover an indefinite number of cases of kinematic equilibrium by assuming some function, in general finite and continuous, which satisfies Laplace's Equation, and then taking as a conductor one inside which the given function is everywhere finite, and which is bounded by surfaces over each of which either the function is constant or its normal derivative zero.

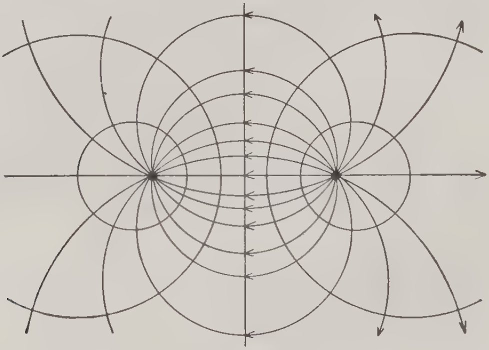

Fig. 59.

If we transform Equation 199 to orthogonal curvilinear coördinates defined by the scalar point functions u, v, w, where w satisfies Laplace's Equation, and assume V to be expressible as a function of w only, we shall obtain (see page 182) the equation $D_w{}^2 V + D_w V \cdot D_w k / k = 0$. If the specific conductivity of a body occupying the space T mentioned at the beginning of the section were not constant but a given function of w, this equation would determine V.

III. ELECTROMAGNETISM.

77. Electromagnetism. Straight Currents. If a steady electric current be sent through a long straight wire, the space in the neighborhood of the current becomes a field of magnetic force. If the medium about the conductor is homogeneous, the direction of the field is such that a small magnetic needle freely suspended by its centre tends to set itself perpendicular to the wire and to the perpendicular dropped from the point of suspension upon the wire, so that "if a person be imagined as swimming in the current which flows from his feet to his head, and if he face the needle, the north pole will be turned towards his left hand." The field is symmetrical about the wire and, according to the rule just given, its direction at any point is normal to the plane drawn through the point and the wire, so that the lines of force are circumferences forming right-handed whirls about the current. To investigate the law of the change of the intensity of the force with the distance from the wire, we may imagine a rigid frame free to turn about the vertical wire as a hinge, and suppose a magnet to be rigidly attached to this frame. It will be found that in this case the frame will have no tendency to rotate under the action of the electromagnetic forces, so that the sum of the moments about the wire, of the forces which the field exerts upon the magnet, must be zero. If r_1 and r_2 are the distances of the poles from the wire, and if $F(r)$ is the intensity of the field at a distance r from the wire, the equality of moments shows that, however the magnet be placed on the frame,

$$r_1 \cdot F(r_1) = r_2 \cdot F(r_2),$$

or, in general, $r \cdot F(r) =$ a constant, k. The value of k is found to be dependent upon the strength, C, of the current in the wire, and can be used to define this strength. We may write, therefore, $F(r) = A \cdot C / r$, where A is a constant depending upon the units in which C is measured. If we use the absolute electromagnetic c.g.s. units, defined below, in determining C, it will presently appear that A is 2.

If we take the plane of the paper for the xy plane, and imagine the wire which carries the current to cut the paper normally at the origin, then, if the current comes from below, the components of the field at the point (x, y) are

$$X = -2\,C \sin(x, r)/r \text{ and } Y = 2\,C \cos(x, r)/r,$$

or $$X = -2\,Cy/(x^2 + y^2) \text{ and } Y = 2\,Cx/(x^2 + y^2).$$

Here $D_y X = D_x Y$ and the magnetic force is, in general, a lamellar vector, so that it has a potential function which, since the lines of force are closed, must be multiple-valued. This potential function is evidently

$$\pm 2\,C \tan^{-1}(y/x) + \text{constant, or } \pm 2\,C\theta + \text{constant},$$

and it satisfies Laplace's Equation. The plus or the minus sign is to be chosen according as we wish to use the derivative of the potential function taken in any direction, or its negative, as a measure of the component of the field in that direction. The line integral of the tangential component of the force taken around any curve in the xy plane which surrounds the origin is $4\pi C$, so that we infer from Stokes's Theorem that at the origin the magnetic force is not lamellar. If a magnetic pole of strength m be moved around any closed path, the work done on it by the magnetic field will be $4\pi m C$ if the path link right-handedly once with the wire, or zero if the path do not link with the circuit. These results are found to be independent of the inductivity of the homogeneous medium about the wire.

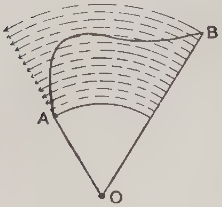

FIG. 60.

Since $D_x X + D_y Y = 0$, the force in the medium about the wire is solenoidal, and the whole flux of force from within outward through any closed surface is zero. If two straight lines parallel to the wire are distant a and b centimetres from it respectively, the flux of force (Fig. 60) through the unit length of any cylindrical surface bounded by the lines is $2\,C \cdot \log(b/a)$. Since we have assumed that a finite quantity of electricity is carried by a conductor of zero cross-section, it is not surprising that this useful analytic result becomes infinite if either a or b is zero.

If two infinitely long straight wires parallel to the z axis carry equal steady currents of strength C in opposite directions, and if they cut the xy plane at the points A_1, A_2, which have the coördinates $(a, 0)$, $(-a, 0)$ respectively, the scalar potential function, Ω, of the field has at the point (x, y, z) the value

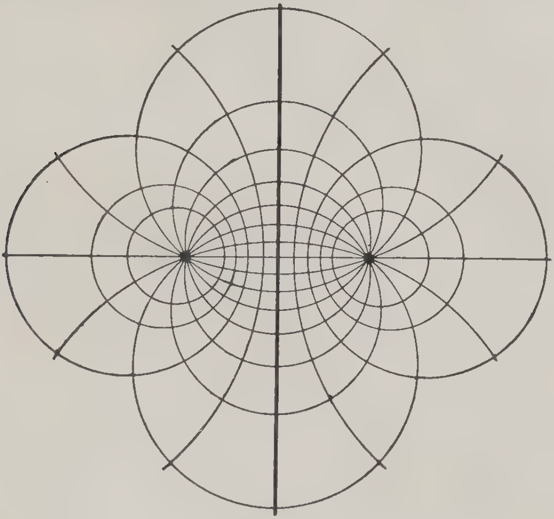

FIG. 61.

$$2\,C \cdot \tan^{-1}[y/(x-a)] - 2\,C \cdot \tan^{-1}[y/(x+a)],$$

or

$$2\,C \cdot \tan^{-1}[2\,ay/(x^2+y^2-a^2)].$$

The conjugate function, Φ, is $+2\,C \cdot \log(r_1/r_2)$, where r_1 and r_2 are the distances of the point (x, y, z) from A_1 and A_2

respectively. The lines of force and the traces in the xy plane of the equipotential surfaces are shown in Fig. 61.

$D_x\Omega = - D_y\Phi$, $D_y\Omega = D_x\Phi$, and the derivative of Ω at any point in the xy plane taken in any direction in the plane is equal to the derivative of Φ at the same point taken in a direction in the plane at right angles to the first. If, then, a curve c is the trace in the xy plane of a cylindrical surface S, the generating lines of which are parallel to the z axis, and if n represents a direction in the plane perpendicular to c, the line integral of $D_n\Omega$ taken along c represents the flux of magnetic force across S per unit of its height, perpendicular to the xy plane. This integral is equal to the line integral of the tangential derivative of Φ along c or to the difference between the values of Φ at the ends of the curve. If this difference is nothing, the corresponding flux is nothing; if Φ is constant all along c, this curve is a line of force.

From the results just obtained, it is evident that if two straight lines parallel to the z axis cut the xy plane in the points B_1, B_2 respectively, the flux of magnetic force through a cylindrical surface bounded by these lines, per unit of its length, parallel to the z axis, is

$$2\, C \cdot \log\left[(A_1B_1 \cdot A_2B_2) / (A_2B_1 \cdot A_1B_2)\right].$$

This represents the flux of force per unit of its height, through a circuit s_2, consisting essentially of two infinitely long straight wires, parallel to the z axis, cutting the xy plane at B_1, B_2 when the steady current C traverses the circuit s_1, consisting essentially of the two wires already mentioned, which cut the xy plane at A_1 and A_2. Symmetry shows that this expression would also give the flux through s_1, due to a steady current C in s_2.

If an infinitely long cylindrical conductor, the generating lines of which are parallel to the z axis, and which is surrounded by a homogeneous medium, carry a steady current in the direction of its length, and if the current density at the point (x', y', z') be q', a function of x' and y' but not of z', the intensity of the magnetic field H within or without the conductor can be obtained by imagining the conductor made up of separate current filaments, each of which has a field like that about a fine straight wire, unaltered by the presence of the others. If L, M, N are the intensities of the components of H parallel to the coördinate axes, L and M are functions of x and y while N is zero.

$$L = -\int\!\!\int \frac{2\,q'(y-y')\,dx'dy'}{(x-x')^2+(y-y')^2},\quad M = \int\!\!\int \frac{2\,q'(x-x')\,dx'dy'}{(x-x')^2+(y-y')^2},$$

where the double integrals extend over the section of the conductor made by the xy plane. If the whole amount of current in the conductor is C, and if u represents the distance of the point (x, y, z) from the axis of z, and ϕ the angle $\tan^{-1}(y/x)$, uL and uM approach the limits $-2\,C\cdot\sin\phi$ and $2\,C\cdot\cos\phi$ when u increases without limit. The line integral of the tangential component of the field, taken around any curve, which surrounds the conductor, is equal to the corresponding integral taken around a circle in the xy plane of infinite radius, with centre at the origin. The value of this last integral is obviously $4\pi C$. Except for points in the mass of the conductor, the integrands of the expressions for L and M are continuous functions of x and y for all values of x' and y' within the limits of integration, and $D_yL = D_xM$ and $D_xL + D_yM = 0$.

At all points in empty space near the conductor, therefore, the field is solenoidal and lamellar and there is a potential function

$$\pm\int\!\!\int 2\,q'\tan^{-1}[(y'-y)/(x'-x)]\,dx'dy',$$

which satisfies Laplace's Equation.

In the special case where the conductor is in the form of a right circular cylinder (or of concentric shells bounded by cylindrical surfaces of revolution), and where the current density is a function only of the distance from the z axis, which coincides with the axis of the conductor, the field is evidently symmetrical, and the direction of the force at any point is perpendicular to the perpendicular to the axis drawn through the point. Everywhere in empty space in the vicinity of the conductor a potential function, Ω, exists, and, since $D_r\Omega = 0$, Laplace's Equation degenerates into $D_\theta^2\Omega = 0$, or $\Omega = a\theta + b$. The work done by the field when a magnetic pole of strength m moves around a circumference, the axis of which is the z axis, is evidently equal to $\pm 4\pi Cm$, where C is the sum of the currents in all the current filaments which the path encloses. Since the line integral of $D_s\Omega$ taken around any such path in empty space in right-handed direction around the current is $2\pi a$, a is equal in absolute value to twice the whole current carried by so much of the conductor as lies within the path. If the direction of the z axis is such that, if the eye is in the positive x axis looking at the origin, a counter-clockwise rotation of the positive axis of y through 90° would make it coincide with the positive z axis, and if $\Omega = -2\,C\theta + b$, the force at any point not in the mass of the conductor, in any direction, is the derivative of Ω at that point taken in the direction in question, and the resultant force is $-D_\theta\Omega / r$ or $2\,C / r$. This is the same as if all the current nearer the z axis than the point in question were flowing through a fine wire coincident with the axis of z. If the infinitely long cylindrical conductor is a uniform tube, the axis of which is the z axis, $\Omega = a\theta + b$ in the empty space within the tube, and, since (on account of symmetry) the resultant force a / r must vanish on the z axis, a is zero and the intensity of the field within the tube is everywhere zero.

We may easily find the intensity of the electromagnetic force at any point P within an infinitely long, round conductor carrying, in the direction of its length, a steady current with intensity the same at all points equally distant from the axis of the conductor, if we imagine a cylindrical surface, S, of revolution coaxial with the conductor drawn through P. The magnetic force at P, due to so much of the current as lies outside S, is nothing; the force due to so much of the current as lies within S is evidently the same as if this portion of the current were concentrated in the axis. If, therefore, a straight conductor in the form of an infinitely long cylinder of revolution of radius a carries a steady current C in the direction of its length, and if the intensity (q) of the current is a function only of the distance (r) from the axis of the conductor, the intensity of the magnetic force (H) is $2\,C/r$ without the cylinder and $\frac{4\,\pi}{r}\int_0^r xq\,dx$ within. The flux of induction per unit length of the cylinder across so much of any plane through the axis as lies within the conductor is

$$Q = 4\,\pi\int_0^a \mu\frac{dr}{r}\int_0^r xq\,dx.$$

If q does not involve r, the current is uniformly distributed through the conductor, the strength of the field within the cylinder is $2\,Cr/a^2$, and Q is equal to μC. If the axis of the cylinder is the z axis, the force components at any inside point distant r from the axis are $L = -2\,Cy/a^2$, $M = 2\,Cx/a^2$, so that H is solenoidal, as it would be if q were any analytic function of r. Since H is not lamellar within the conductor, it is at the outset clear that there can be no scalar potential function Ω there; it is well to notice, however, that, if the derivative of a scalar function, Ω, at any point in any direction were required to show the force at that point in the given direction, it would need to satisfy, within the conductor, the two incompatible conditions,

$$D_r\Omega = 0, \quad (D_\theta\Omega)/r = -2\,Cr/a^2.$$

Since H is solenoidal even at inside points, we may ask whether its components are not the components of some vector, Q, which may be regarded as a vector potential function of H, and it is clear that a vector of intensity $-\pi qr^2$, directed at every point parallel to the z axis, satisfies all the conditions, as do many other vectors. The component, at any point within the conductor, in any direction, of the curl of the vector $(0,\ 0,\ -\pi q r^2)$ shows the component of the magnetic force H at the point in the given direction. The abscissas of Fig. 62 represent distances from the axis of the conductor, and the ordinates the corresponding values of the resultant magnetic force in the case just considered.

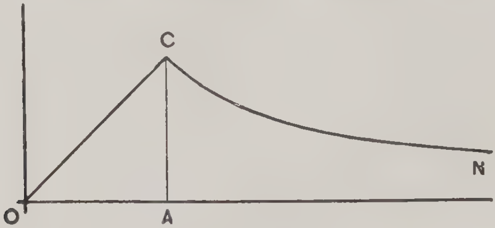

Fig. 62.

If a uniformly distributed current C be brought up normally through the plane of the paper by an infinitely long cylinder of revolution and down through a similar cylinder parallel to the first, the lines of force without the cylinders are of the same shape as those shown in Fig. 61. The curve in Fig. 63 shows the intensity of the field at points in a straight line which cuts the axes of the cylinders perpendicularly.

If two infinitely long, coaxial, cylindrical surfaces of revolution carry symmetrically equal and opposite currents, each of strength C, parallel to their common axis, the space between the surfaces is a field of electromagnetic force of strength $2\,C/r$, where r is the distance from the axis. There is no force within the inner surface or without the outer one.

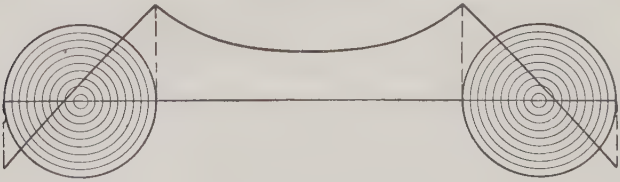

Fig. 63.

In the case of a long, straight wire of radius a surrounded by a coaxial tube of radii b and c, and carrying uniformly distributed a steady current C which returns through the tube, the electromagnetic force is evidently zero on the axis of the wire and continuous at every distance r from the axis. If w_1 and w_2 are the intensities of the current in the wire and in the tube respectively, $C = w_1 \pi a^2 = w_2 \pi (c^2 - b^2)$, and if we apply the formulas just proved, we shall learn that the strengths of the fields within the wire, between the wire and the tube, in the body of the tube and without the tube, are given by the expressions $2\pi w_1 r$, $2\pi a^2 w_1 / r$, $2\pi w_2 (c^2 - r^2)/r$, and 0.

It is to be noted that the strength of the magnetic field due to a given electric current is, in the homogeneous medium which surrounds the current, wholly independent of the permeability of this medium, whereas the field due to a given magnet would be inversely proportional to the inductivity. If the fields of a given circuit and a given magnet were the same in one homogeneous medium, they would not be the same in another homogeneous medium of different magnetic inductivity. The induction due to a current circuit in a homogeneous medium filling all space is proportional to the inductivity, as is the energy in the medium. The induction due to magnetic matter surrounded by a homogeneous medium is independent of the inductivity of the medium. The action of a distribution of magnetic matter in an infinite homogeneous medium on a circuit carrying a steady current is not altered by changing the inductivity of the medium.

78. Closed Circuits. Experiment shows that if a steady current of strength C runs in a simple linear circuit of any form, there is a magnetic field in the neighborhood of the conductor and the lines of the field are all linked right-handedly with the circuit. If a unit magnetic pole be carried round any closed path which does not link with the circuit, the work done by the field on the pole is zero, whatever the character of the medium near the circuit, so that a potential function exists in the so-called empty space about the wire. This potential must be multiple-valued, since the lines of force are closed. If the pole be carried round a closed path which links once with the circuit, the work done on the pole by the field is $\pm 4\pi C$, whether the medium intersected by the path is homogeneous or not. We infer, therefore, that no scalar potential function exists in the wire which carries the current.

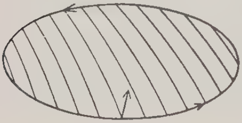

FIG. 64.

It follows from the experiments of Ampère that the field of magnetic *force*, due to a steady current of C electromagnetic units flowing in a closed linear circuit in a homogeneous medium, is identical with the field of magnetic *induction* due to a simple magnetic shell (Fig. 64) of strength C bounded by the circuit. This statement defines the electromagnetic unit of current. The magnetic force, due to a current of C electromagnetic units flowing in a closed linear circuit in a homogeneous medium of inductivity μ, is the same in magnitude and direction at any point P as the force due to a simple magnetic shell of strength $C\mu$ bounded by the circuit. The shell may be of any form, provided that it does not pass through P and that its positive side is such that the current surrounds right-handedly the direction of polarization. To make the potential function single-valued, we may cover the circuit by a cap or diaphragm, fix at pleasure the value Ω_O of the potential function at some one point O in the field, and define the value at any other point Q to be the line integral of the magnetic force taken from O to Q along any path which does not cut the diaphragm.

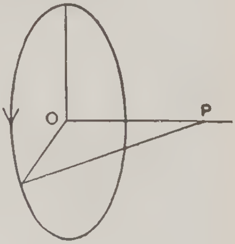

FIG. 65.

At any point P on the axis of a circular current of radius a, at a distance x from the plane of the circuit, the circuit subtends the solid angle

$$\omega = 2\pi(1 - \cos\theta) = 2\pi(1 - x/\sqrt{a^2 + x^2}).$$

If the strength of the current in the circuit is C, the magnetic force at P is directed along the axis of the circuit (Fig. 65) and is numerically equal to the negative of the derivative with respect to x of $C\omega$. The intensity of the force is, therefore,

$$2\pi a^2 C/(x^2 + a^2)^{3/2}$$

and at the centre of the circuit, where $x = 0$, it is $2\pi C/a$. This result evidently agrees with the awkward statement sometimes used to define the electromagnetic unit of current. "If one centimetre of a linear circuit which carries the unit current be bent into an arc of one centimetre radius, the strength of the field at the centre of the arc, due to this portion of the circuit, will be one dyne." The ampere, which is the practical unit of current intensity, is one-tenth of the unit just defined.

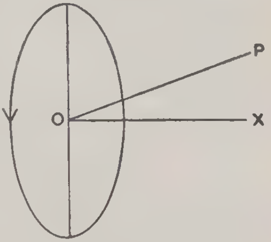

Fig. 66.

If for convenience we denote the quantity a/x by u and its reciprocal by v, the potential function $(C\omega)$ just found may be written in either of the forms $2\pi C\{1 - 1/\sqrt{1 + u^2}\}$ or $2\pi C\{1 - v/\sqrt{1 + v^2}\}$, and, according as x is greater or less than a, we may use one or other of the developments

$$2\pi C\left\{\frac{1}{2}u^2 - \frac{1\cdot 3}{2\cdot 4}u^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}u^6 - \cdots\right\},$$

$$2\pi C\left\{1 - v + \frac{1}{2}v^3 - \frac{1\cdot 3}{2\cdot 4}v^5 + \cdots\right\}.$$

If, then, P_1, P_2, P_3, $\cdots$ represent zonal harmonics expressed in terms of $\cos\alpha$, and if u_1 and v_1 represent a/r and r/a, the value of the potential function at a point distant r from the centre of the circuit, in a direction (Fig. 66) making an angle a with the x axis, is given according as r is greater or less than a by one or other of the developments

$$2\,\pi C\left\{\frac{1}{2}\,u_1^{\,2}\cdot P_1-\frac{1\cdot 3}{2\cdot 4}\,u_1^{\,4}\cdot P_3+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\,u_1^{\,6}\cdot P_5\cdot\cdot\cdot\right\},$$

$$2\,\pi C\left\{1-v_1\cdot P_1+\frac{1}{2}\,v_1^{\,3}\cdot P_3-\frac{1\cdot 3}{2\cdot 4}\,v_1^{\,5}\cdot P_5+\cdot\cdot\cdot\right\}.$$

If an infinitely long straight wire which carries a steady current, C, forms part of a plane closed circuit, all the other parts of which are at infinity, and if the plane of the circuit be used as the xz plane and the wire as the z axis, the solid angle subtended at the point (x, y, z) by the circuit is $2(\pi-\theta)$, where $\tan\theta = y/x$. The force components at the point are, then, the negatives of the derivatives with respect to x and y respectively of $2\,C(\pi-\theta)$, that is, $-2\,Cy/(x^2+y^2)$ and $+\,Cx/(x^2+y^2)$, as we already know.

79. The Law of Laplace. Mechanical Action on a Conductor which carries a Current in a Magnetic Field. It will be evident from the discussion on page 218 that the strength of the magnetic field, H, due to a steady current of C electromagnetic units in a rigid linear circuit may also be computed, whatever the inductivity of the homogeneous surrounding medium, on the assumption that every element ds of the circuit (Fig. 67) makes a contribution numerically equal to

$$C\cdot\sin(r, ds)\cdot ds/r^2,$$

to the force at a point P, where r is the distance of ds from P. The direction of the contribution is normal to the plane of P and ds, and such that a north magnetic pole at P tends to whirl right-handedly about a straight line drawn through ds in the direction of the current. For a simple illustration of the use of this rule, which is sometimes called "Laplace's Law," let P be a point at a distance r_0 from an infinitely long straight wire which carries a current C, and let s be the distance of ds from the foot of the perpendicular dropped from P upon the wire. If the angle (r, ds) be denoted by θ, $s = r_0 \operatorname{ctn} \theta$, $ds = -r_0 \csc^2 \theta \, d\theta$, $r = r_0 \csc \theta$. All the elements of the current conspire to produce at P a magnetic force perpendicular to the plane of P and the wire. The magnitude of this force is

$$C \int_{-\infty}^{+\infty} \frac{\sin \theta \cdot ds}{r^2} = -\frac{C}{r_0} \int_{\pi}^{0} \sin \theta \, d\theta = \frac{2\,C}{r_0},$$

as before.

If a circuit is not plane, the different elements of the current will contribute to the magnetic force, at a point P, elementary forces which do not all have the same directions. In this case it is necessary to compute separately the components L, M, N of H. If the coördinates of the beginning of ds are x_1, y_1, z_1, and those of the end $x_1 + dx_1$, $y_1 + dy_1$, $z_1 + dz_1$, while those of P are x, y, z, the direction cosines of r and ds are $(x_1 - x)/r$, $(y_1 - y)/r$, $(z_1 - z)/r$, and dx_1/ds, dy_1/ds, dz_1/ds, and, if the direction cosines of dH, the contribution to the force at P made by the current element ds, are l, m, n, then, since this direction is perpendicular to r and to ds,

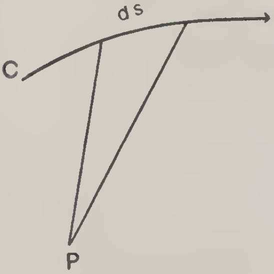

FIG. 67.

$$l(x_1 - x) + m(y_1 - y) + n(z_1 - z) = 0,$$
$$l\,dx_1 + m\,dy_1 + n\,dz_1 = 0,$$
$$l^2 + m^2 + n^2 = 1.$$

If we represent the expressions $(y_1 - y)\,dz_1 - (z_1 - z)\,dy_1$, $(z_1 - z)\,dx_1 - (x_1 - x)\,dz_1$, $(x_1 - x)\,dy_1 - (y_1 - y)\,dx_1$ by δ', δ'',

δ''' respectively, and $\delta'^2 + \delta''^2 + \delta'''^2$ by δ, we learn from these equations that $l = \delta'/\delta$, $m = \delta''/\delta$, $n = \delta'''/\delta$,

$$\cos(r_1, ds) = [(x_1 - x)\,dx_1 + (y_1 - y)\,dy_1 + (z_1 - z)\,dz_1]/r\,ds,$$

and $\sin(r, ds) = \delta/r\,ds$.

If, then, the components of dH are dL, dM, dN, we have the equations $dL = C\delta'/r^3$, $dM = C\delta''/r^3$, $dN = C\delta'''/r^3$, and from these, by integration over the circuit, the force at P may be computed.

Since action and reaction are equal and opposite, a unit magnetic pole at P would exert upon the element ds of the conductor which carries the current a mechanical or "ponderomotive" force the components of which would be $-C\delta'/r^3$, $-C\delta''/r^3$, $-C\delta'''/r^3$. These components, written in terms of the components

$$L_m = (x_1 - x)/r^3,\ M_m = (y_1 - y)/r^3,\ N_m = (z_1 - z)/r^3,$$

of the magnetic field at ds due to the pole at P, are

$$C(N_m dy_1 - M_m dz_1),\ C(L_m dz_1 - N_m dx_1),\ C(M_m dx_1 - L_m dy_1),$$

and, since so far as this force is concerned the origin of the magnetic field is immaterial, these expressions give the components of the mechanical force which act upon the element ds of a circuit carrying a steady current C in any magnetic field which at ds has the components L_m, M_m, N_m.

If the magnetic field at ds_1 — an element of a linear circuit s_1 which carries a steady current C_1 — is due to a steady current C_2 in another circuit s_2, the element ds_2 of the second circuit at the point (x_2, y_2, z_2) contributes to the magnetic field at ds_1 at the point (x_1, y_1, z_1) components numerically equal to

$$\begin{aligned}&\frac{C_2}{r^3}[(z_1 - z_2)\,dy_2 - (y_1 - y_2)\,dz_2],\\ &\frac{C_2}{r^3}[(x_1 - x_2)\,dz_2 - (z_1 - z_2)\,dx_2],\\ &\frac{C_2}{r^3}[(y_1 - y_2)\,dx_2 - (x_1 - x_2)\,dy_2],\end{aligned} \qquad [202]$$

so that the x component of the mechanical force exerted upon the circuit element ds_1 by the circuit element ds_2 is

$$dX_1 \equiv \frac{C_1 C_2}{r^3} \{[(y_1 - y_2)\,dx_2 - (x_1 - x_2)\,dy_2]\,dy_1 - [(x_1 - x_2)\,dz_2 - (z_1 - z_2)\,dx_2]\,dz_1\},$$

or
$$C_1 C_2 \cdot D_{x_1}(1/r)\,[dx_1 \cdot dx_2 + dy_1 \cdot dy_2 + dz_1 \cdot dz_2] - C_1 C_2\,dx_2\,[D_{x_1}(1/r)\,dx_1 + D_{y_1}(1/r)\,dy_1 + D_{z_1}(1/r)\,dz_1],$$

or
$$\frac{C_1 C_2\,ds_1\,ds_2}{r^2}[\cos(x, r) \cdot \cos(ds_1, ds_2) - \cos(x, ds_2)\cos(r, ds_1)], \qquad [203]$$

where r is the distance of ds_2 from ds_1.

The x component, X_1, of the whole mechanical force exerted upon the rigid circuit s_1 by the rigid circuit s_2 is to be found by integrating the expression just found over both circuits.

The resulting integral will evidently not be changed if we add to the integrand any quantity which disappears when integrated about either circuit, and this fact makes it possible to find many other expressions * for the mechanical force exerted upon an element of one circuit by an element of another, which will account mathematically for the observed forces between two rigid closed circuits.

According to Ampère's analysis, the resulting action between the two elements ds_1, ds_2 is an attraction in the line joining them of intensity

$$\frac{C_1 C_2\,ds_1\,ds_2}{r^2}[2\cos(ds_1, ds_2) - 3\cos(r, ds_1) \cdot \cos(r, ds_2)].$$

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* For exhaustive treatments of this important subject the reader should consult Ampère, *Gilbert's Ann.*, 1821; Ampère, *Mém. de l'Académie*, 1823, 1827; W. Weber, *Ges. Werke;* Grassmann, *Pogg. Ann.*, 1845; F. E. Neumann, *Abh. Berl. Akademie*, 1845; Wiedemann, *Lehre von der Elektricität;* Maxwell, *Treatise on Electricity and Magnetism*, §§ 502–527; Webster, *Theory of Electricity and Magnetism*, §§ 217–221. For convenience of reference I have followed Professor Webster's order, and in part his notation in the brief treatment of the Electrodynamic Potential given in Section 80. See Problem 307, page 452.

On this assumption two elements in the same straight line repel each other with a force $C_1C_2ds_1ds_2/r^2$, while two parallel elements perpendicular to the line which joins them attract each other with a force $2\,C_1C_2ds_1ds_2/r^2$. These expressions, like those which precede, hold good whether the elements ds_1, ds_2 belong to the same circuit or to two different circuits.

If two infinitely long straight wires (s_1, s_2), parallel to each other at a distance a apart, carry in the same direction the steady currents C_1, C_2 respectively, the mechanical force exerted on s_1 by s_2 is evidently $C_1C_2\int_1\int_2[\cos(x,r)/r^2]\,ds_1\cdot ds_2$, or $(2\,C_1C_2/a)\int ds_1$, so that every unit length of s_1 is attracted towards s_2 with a force of $2\,C_1C_2/a$ dynes.

If each of two closed circuits (s_1, s_2) which carry steady currents, C_1, C_2, consists essentially of two infinitely long wires parallel to the z axis, if the currents come up through the xy plane in the two circuits at the points $(0, a)$, (c, b) respectively, and go down at the points $(0, -a)$, $(c, -b)$, the first circuit experiences a force tending to urge it in the direction of the x axis, and the intensity of this force per unit length of both wires of s_1 is $4\,cC_1C_2\{1/[(a-b)^2+c^2]-1/[(a+b)^2+c^2]\}$.

It is evident from the discussion of the properties of magnetic shells in air given on page 217 that the mechanical action on a rigid linear circuit carrying a steady current C in a magnetic field (caused either by permanent magnets or by other currents or by both) may be mathematically accounted for on the supposition that every element ds of the circuit is urged by a force equal to $C\,ds$ times the component (F), perpendicular to ds, of the total magnetic induction. The direction D of this elementary force is perpendicular to the plane of C and F in the sense shown in Fig. 68.

The same assumption will account for the phenomena observed when a deformable circuit is placed in a magnetic field.

According to this theory the component in any direction u of the force on the element ds is $C ds \cdot B \cdot \sin(B, ds) \cos \alpha$, where α is the angle between u and the normal to the plane of B and ds, and this is numerically equal to the volume of a parallelopiped, adjacent edges of which are represented in magnitude and direction by $C ds$, B, and a unit length in the direction u. This volume may also be represented by $C ds \cdot \sin(u, ds) \cdot B'$, where B' is the component of the induction B, normal to the plane of u and ds, and this expression for the force component is occasionally useful.

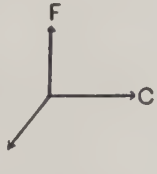

FIG. 68.

If (l, m, n) are the direction cosines of the element ds and if the components of the induction B are B_x, B_y, B_z,

$$\begin{aligned}\sin(B, ds) = \{(m \cdot B_z - n \cdot B_y)^2 + (n \cdot B_x - l \cdot B_z)^2 \\ + (l \cdot B_y - m \cdot B_x)^2\}^{\frac{1}{2}} / \{B_x^2 + B_y^2 + B_z^2\}^{\frac{1}{2}}\end{aligned}$$

and the resultant electromagnetic force on the circuit element ds has the value

$$C\{(m \cdot B_z - n \cdot B_y)^2 + (n \cdot B_x - l \cdot B_z)^2 + (l \cdot B_y - m \cdot B_x)^2\}^{\frac{1}{2}} \cdot ds.$$

If ds is an element of a current filament of cross-section ω in a massive conductor in which the current vector is q or (u, v, w), we have $q\omega = C$, $u\omega = lC$, $v\omega = mC$, $w\omega = nC$, and the electromagnetic force may be written

$$\omega\{(v \cdot B_z - w \cdot B_y)^2 + (w \cdot B_x - u \cdot B_z)^2 + (u \cdot B_y - v \cdot B_x)^2\}^{\frac{1}{2}} \cdot ds.$$

The components parallel to the coördinate axes of the electromagnetic force per unit volume of the conductor are, therefore, $(v \cdot B_z - w \cdot B_y)$, $(w \cdot B_x - u \cdot B_z)$, $(u \cdot B_y - v \cdot B_x)$.

If the element ds be moved parallel to itself through the distance du, the mechanical work done on it by the forces of the field can be represented numerically by the volume of a parallelopiped, conterminous edges of which are $C\,ds$, B, and du; this volume is numerically equal to C times the number of lines of induction of the field cut by the element during the translation. If an observer be imagined to lie in the element in such a way that the current enters at his feet and goes out at his head, and if he faces in such a direction that he can look along the lines of force, the work done by the translation will be positive if these lines appear to pass him from left to right, that is, if the displacement is to his left. It is easy to see, moreover, that if the element ds be revolved about any axis through a small angle, the work done upon it may be represented by C times the number of lines of induction cut by the element during the displacement; we may infer, therefore, that the electromagnetic work done by the field upon any portion s of a circuit during any displacement is measured by the product of the current strength and the number of lines of induction cut by s. The direction in which a rigid closed linear circuit carrying a steady current C in a magnetic field of any kind will tend to move may be inferred from the fact that the circuit will behave in this respect like the equivalent magnetic shell.

It is easy to see from the discussion on page 216 that the mutual potential energy of an external field and the magnetic shell mechanically equivalent to a given circuit, — that is, the mechanical work that must be done to bring the shell already formed into the field, — is equal to $-CN$, where N is the whole number of lines (unit tubes) of induction of the field which the current surrounds right-handedly. The circuit will tend to move, therefore, so as to make N as large as possible. If, for instance, a plane circuit of area A carries a steady current C in a uniform field of induction of intensity B, any motion of the circuit parallel to itself would not change the induction through it, and there is no tendency to any such motion; if the normal to the plane of the circuit makes an angle θ with the direction of the field, a couple, of moment $C \cdot A \cdot B \cdot \sin\theta$, acts on the circuit and tends to decrease θ.

If into a magnetic field F_0 which has the components X_0, Y_0, Z_0 a linear circuit carrying a steady current be introduced, and if the electromagnetic field due to the current alone is F_1, or (X_1, Y_1, Z_1), the whole field is $(X_0 + X_1, Y_0 + Y_1, Z_0 + Z_1)$, and the whole magnetic energy in the field is

$$\frac{1}{8\pi}\iiint_\infty \mu\{(X_0 + X_1)^2 + (Y_0 + Y_1)^2 + (Z_0 + Z_1)^2\}\, d\tau,$$

or

$$\frac{1}{8\pi}\iiint_\infty \mu(X_0^2 + Y_0^2 + Z_0^2)\, d\tau$$

$$+ \frac{1}{8\pi}\iiint_\infty \mu(X_1^2 + Y_1^2 + Z_1^2)\, d\tau$$

$$+ \frac{1}{4\pi}\iiint_\infty \mu(X_0X_1 + Y_0Y_1 + Z_0Z_1)\, d\tau.$$

The first integral is the magnetic energy of the original field, the second that of the field of the circuit alone, and the third the magnetic energy due to the introduction of the circuit when formed into the field. We may now show that this last term, which may be written

$$\frac{1}{4\pi}\iiint_\infty \mu F_0 \cdot F_1 \cdot \cos(F_0, F_1)\, d\tau,$$

is equal to the product of the strength of the current and the flux of induction of the original field in the positive direction through the circuit. Since all the equipotential surfaces of the field F_1 are bounded by the circuit, we may cap the circuit by a whole series* of such surfaces and write the total induction through the circuit due to the outside field in the form

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* A. Gray, *Treatise on Magnetism and Electricity*, Vol. I, p. 293.

$$M \equiv \int\int \mu (lX_0 + mY_0 + nZ_0)\, dS \equiv \int\int \mu F_0 \cdot \cos(F_0, F_1)\, dS,$$

where the integration is to be taken over any one of these caps and where l, m, n are the direction cosines of the normal to the cap.

If a unit magnetic pole were carried around any line of force s_1 of the field F_1, the work done on it would be 4π times the current C in the circuit, so that $4\pi C = \int F_1 \cdot ds_1$. If we multiply each side of this last equation by M, we have

$$C \int\int \mu (lX_0 + mY_0 + nZ_0)\, dS$$
$$= \frac{1}{4\pi} \int\int \mu F_0 \cdot \cos(F_0, F_1)\, dS \cdot \int F_1\, ds_1.$$

Since the caps are equipotential, $F_1\, ds_1$ has the same value for all lines of force between any two caps, and since the induction μF_0 is solenoidal, the first integral factor of the second member has the same value for all the caps. We may find the value of the second member, therefore, by imagining space divided up into elements which are portions of tubes of force of the field F_1 bounded by equipotential surfaces of this field, multiplying the volume of each element by the value in it of $\mu F_0 \cdot F_1 \cdot \cos(F_0, F_1)$, and finding the limit of the sum of all these quantities divided by 4π. The value of the volume integral must be, however, independent of the shapes of the elements, and we have, in general,

$$C \int\int \mu (lX_0 + mY_0 + nZ_0)\, dS$$
$$= \frac{1}{4\pi} \int\int\limits_{\infty}\int \mu \cdot F_0 \cdot F_1 \cdot \cos(F_0, F_1)\, d\tau.$$

The magnetic energy in the medium is often called the "electrokinetic energy." That portion of the electrokinetic energy which is due to the introduction of the circuit already established into the given field is evidently the negative of the mutual potential energy, corresponding to work done against mechanical forces, of the equivalent magnetic shell and the field.

If a portion *s* of a circuit electrically connected through mercury cups with the rest of the circuit, which is fixed, be rotated and finally brought back to its original position, electromagnetic work will be done on *s* if it cut lines of the field in positive direction during the motion, but the whole circuit may be represented by the same magnetic shell at the beginning and at the end of the process, and the mutual potential energy of the circuit and the field is unaltered by the displacement. Under these circumstances, as will appear in the sequel, currents are induced in *s* by the motion.

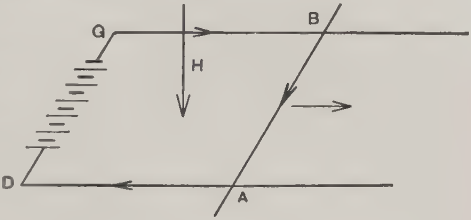

FIG. 69.

If in the case of the circuit shown in Fig. 69 the conductor *AB* is free to slide on the rails *DA*, *GB* in such a way as to be always parallel to *DG*, it will move in the direction indicated by the detached arrow, the circuit will be made to embrace in the positive direction a greater number of lines of induction, and the electrokinetic energy will be increased. If the motion take place without external help, the necessary energy must be furnished at the expense of chemical action in the battery. Let *E* be the electromotive force of the battery, *r* the resistance of the circuit at any instant, and *C* the current which then passes through it: the energy furnished by the chemical action in the battery during the time *dt* will be *ECdt*, and of this a part, at least, $C^2r\,dt$, appears as heat in the conductors which make up the circuit. If AB be held still, C will have such a value, C_0, that $EC_0 = C_0{}^2r$. If, however, AB be moving toward the right, the current will be smaller than C_0, EC will be a fraction of EC_0, C^2r a smaller fraction of $C_0{}^2r$, and EC will, therefore, be greater than C^2r. The difference $(EC - C^2r)\,dt$ now represents the work done during the time dt in moving AB: a part of this work is used in overcoming friction on the rails, a part in communicating kinetic energy to AB, and a third part in increasing the energy of the medium. If for convenience we denote $(EC - C^2r)\,dt$ by $C \cdot dp$, we shall have $E - D_tp = Cr$, and the current is the same as if there were in the circuit an electromotive force D_tp opposed to that of the battery. If an external force were applied to AB tending to move it to the right, the velocity might be increased so much that the current would be reduced to zero or caused to flow in the opposite direction. If, however, AB were forced to move to the left by external forces, the current in the circuit would become greater than C_0 and would have the same direction as E.

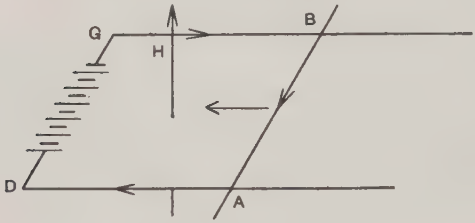

FIG. 70.

Fig. 70 illustrates a case where the resultant magnetic field is, as before, normal to the plane of the circuit, though the field lines thread the circuit in the negative direction; in this case AB will tend to move toward the left.

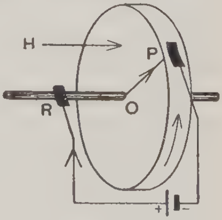

FIG. 71.

Fig. 71 represents Faraday's metal disc, mounted on a metallic arbor and free to turn about a horizontal axis. At any instant the current flows in the disc from the centre to the brush P and the conductor which carries the current is urged to turn in the direction indicated by the arrow. The energy in the medium is not increased by the motion of the disc, and the work done by the battery is spent in heating the conductors in the circuit, in overcoming friction and the resistance of the air, and in increasing the kinetic energy of the disc. If the field is uniform, if S is the area of one face of the disc, and if the media are of unit inductivity, the work done on the disc each turn is CHS, and if it is making n turns per second, we have $EC = C^2r + CHSn$. If the disc be used as a motor to overcome resistance of some kind, and if the energy required per turn is $f(n)$, $CHS = f(n)$, and from these two equations n and C may be found, if f be a known function.

In the arrangement shown in Fig. 72 a rigid wire free to turn about the axis of a fixed vertical magnet makes electrical contact with the magnet at its middle. The current from a battery flows through a circuit made up of the wire, the magnet, and a supplementary fixed conductor forming a prolongation of the axis of the magnet. In this case the wire will turn continuously in the direction indicated. It is easy to show that a fixed magnetic field cannot cause continued rotation of a complete rigid circuit about a fixed axis.

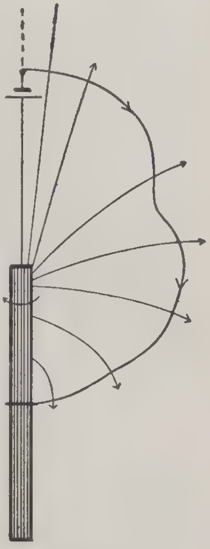

FIG. 72.

80. The Electrodynamic Potential. If while a linear circuit, s_2, which carries a steady current, C_2, remains fixed, a neighboring linear circuit, s_1, which carries a steady current, C_1, is deformed or moved without being stretched, so that every element ds_1 is unchanged in length but the coördinates of the beginning of the element receive increments δx_1, δy_1, δz_1, which are analytic functions of x_1, y_1, z_1, the work done by the forces which s_2 exerts upon s_1 is approximately equal to

$$\int_1\int_2 (dX_1\cdot\delta x_1 + dY_1\cdot\delta y_1 + dZ_1\cdot\delta z_1),$$

or to

$$C_1C_2\int_1\int_2 [D_{x_1}(1/r)\cdot\delta x_1 + D_{y_1}(1/r)\cdot\delta y_1 + D_{z_1}(1/r)\cdot\delta z_1]$$
$$[dx_1\cdot dx_2 + dy_1\cdot dy_2 + dz_1\cdot dz_2]$$
$$- C_1C_2\int_1\int_2 [D_{x_1}(1/r)\cdot dx_1 + D_{y_1}(1/r)\cdot dy_1 + D_{z_1}(1/r)\cdot dz_1]$$
$$[dx_2\cdot\delta x_1 + dy_2\cdot\delta y_1 + dz_2\cdot\delta z_1].$$

The first factor under the integral signs in the second integral of the last expression is equal to $D_{s_1}(1/r)\cdot ds_1$, and if we integrate the whole integrand by parts with respect to s_1, we get

$$[(dx_2\cdot\delta x_1 + dy_2\cdot\delta y_1 + dz_2\cdot\delta z_1)/r]_{\text{taken between limits}}$$
$$-\int_1 (dx_2\cdot d\delta x_1 + dy_2\cdot d\delta y_1 + dz_2\cdot d\delta z_1)/r,$$

where the expression in brackets, having the same value at both limits, can be omitted. The expression for the elementary work done on s_1 during its displacement is, therefore,

$$C_1C_2\int_1\int_2 [D_{x_1}(1/r)\cdot\delta x_1 + D_{y_1}(1/r)\cdot\delta y_1 + D_{z_1}(1/r)\cdot\delta z_1]$$
$$[dx_1\cdot dx_2 + dy_1\cdot dy_2 + dz_1\cdot dz_2]$$
$$+ C_1C_2\int_1\int_2 (dx_2\cdot d\delta x_1 + dy_2\cdot d\delta y_1 + dz_2\cdot d\delta z_1)/r,$$

and this is obviously equal to the variation of the integral

$$C_1C_2\int_1\int_2 (dx_1\cdot dx_2 + dy_1\cdot dy_2 + dz_1\cdot dz_2)/r,$$

caused by the elementary displacement. This last integral written in the form

$$C_1 C_2 \int_1 \int [\cos (ds_1 \cdot ds_2) / r] \, ds_1 \cdot ds_2 \qquad [204]$$

gives what is often called F. E. Neumann's Expression for the *Electrodynamic Potential.* The increase in the value of this function caused by any finite displacement of s_1 evidently represents the work done on s_1 by the field due to s_2 during the displacement: this work depends only upon the original and final configurations. The Electrodynamic Potential corresponds to that portion of the electrokinetic energy which is due to the mutual proximity of the circuits. Its negative is equal to what is sometimes called the mutual potential energy due to the mechanical forces acting between the circuits. It is important to notice that although the ponderomotive forces which urge a rigid circuit carrying a given current, C, in a magnetic field can be correctly found from the expression for the mutual potential energy of the field and a magnetic shell of strength C bounded by the circuit, this may be regarded from one point of view as merely a convenient mathematical device. If the shell were to move under the action of the field alone and acquire kinetic energy and overcome external resistance, this work would be done at the expense of the mutual potential energy of the field and the shell. If, C being kept constant, the circuit were to move under the action of the field in exactly the same way, the work would be done at the expense of the generator which maintains the current. In other words, there is no sensible mutual potential energy of the field and the circuit, the exhaustion of which measures the work done by the forces of the field during any displacement of the circuit.

The integrand in the expression given by Neumann can be increased at pleasure by any quantity which disappears when integrated around either s_1 or s_2. Such a quantity is $\lambda \cdot D_{s_1} D_{s_2} r$, or $\lambda [\cos (r, ds_1) \cdot \cos (r, ds_2) - \cos (ds_1, ds_2)] / r$, where λ is any constant. The corresponding form of the Electrodynamic Potential is

$$C_1C_2\int_1\int_2 \{\lambda \cos(r,\, ds_1)\cdot\cos(r,\, ds_2)\}(1/r)\, ds_1\cdot ds_2$$

$$+\, C_1C_2\int_1\int_2 \{(1-\lambda)\cdot\cos(ds_1,\, ds_2)\cdot(1/r)\}\, ds_1\cdot ds_2.$$

A form sometimes convenient is obtained by putting $\lambda = 1$.

In the case of two vertical, coaxial, circular wire circuits of radii r_1 and r_2, at a distance a apart (Fig. 73), we may

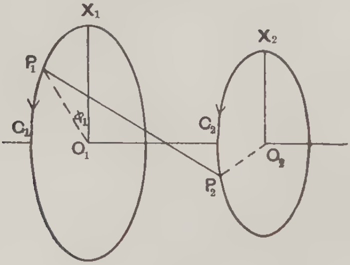

FIG. 73.

denote by ϕ_1 and ϕ_2 the angles which radii, drawn from ds_1, ds_2 respectively to the centres of their circuits, make with the vertical and put $x_1 = r_1\cos\phi_1$, $x_2 = r_2\cos\phi_2$, $y_1 = r_1\sin\phi_1$, $y_2 = r_2\sin\phi_2$, $r^2 = a^2 + r_1{}^2 + r_2{}^2 - 2\,r_1r_2\cdot\cos(\phi - \phi_2)$. The expression

$$P \equiv C_1C_2\int_1\int_2 (dx_1\cdot dx_2 + dy_1\cdot dy_2 + dz_1\cdot dz_2)/r$$

then becomes

$$C_1C_2r_1r_2\int_0^{2\pi} d\phi_2\int_0^{2\pi}\frac{\cos(\phi_1-\phi_2)\,d\phi_1}{\sqrt{a^2 + r_1{}^2 + r_2{}^2 - 2\,r_1r_2\cos(\phi_1-\phi_2)}},$$

or

$$C_1C_2r_1r_2\int_0^{2\pi} Q\,d\phi_2.$$

That the definite integral Q is not a function of ϕ_2 follows from the fact that the definite integral which represents its partial derivative with respect to ϕ_2 is the limit of the sum of elements which destroy each other in pairs: we may therefore give to ϕ_2 in the expression for Q any convenient value (say zero) and write $P = 2\pi\, C_1 C_2 r_1 r_2 Q$. We may conveniently transform the integral which represents Q by putting

$$2\,\theta \equiv \phi_1 - \pi,\ k^2 \equiv 4\,r_1 r_2/[a^2 + (r_1 + r_2)^2],$$

and get

$$P = 2\,\pi k C_1 C_2 \sqrt{r_1 r_2} \int_0^{\pi/2} \frac{(1 - 2\sin^2\theta)\,d\theta}{\sqrt{1 - k^2 \sin^2\theta}},$$

or

$$4\,\pi C_1 C_2 \sqrt{r_1 r_2} \left\{ \left(\frac{2}{k} - k \right) K - \frac{2}{k}\, E \right\},$$

where K and E are the complete elliptic integrals of the first and second kinds. The numerical values of these integrals for various values of k are to be found in "A Short Table of Integrals" (Ginn & Company, Boston). It is to be noted that if in this analysis we imagine finite currents to be carried by conductors of zero cross-section, and r_1 and r_2 to be equal, then, if a approaches zero, k approaches unity and P grows large without limit. The derivative of P with respect to a gives in general the mutual attraction of the two circuits.

If the external field about a linear circuit s_1, carrying a circuit C_1, is due to a current C_2 in another linear circuit s_2, we have two different expressions for the mutual potential energy of the magnetic shells which correspond to the two circuits. These are $- C_1 N$, where N is the number of lines of induction due to C_2 which thread s_1 positively, and the negative of the Electrodynamic Potential of the two circuits. When C_1 and C_2 are both unity the Electrodynamic Potential measures the magnetic induction through either circuit when the unit current traverses the other.

The number of lines of magnetic induction which thread either of two simple linear circuits, made of non-magnetic material and removed from the neighborhood of other currents and permanent magnets, when the unit current passes through the other circuit, is called the *coefficient of mutual induction* or the *mutual inductance* of the two circuits. The numerical value of this coefficient depends upon the character of the media in the neighborhood of the circuit.

If two exactly similar linear circuits, s_1 and s_2, carrying steady currents of unit intensity, lie side by side, and if one of them (s_2) be imagined to move up towards coincidence with the other, the value of the integral which represents the Electrodynamic Potential approaches the form

$$L = \int_1 \int_1 \frac{\cos(ds_1, ds_2)\, ds_1 \cdot ds_2}{r},$$

where the integration is to be extended twice over the same circuit. If the circuits are supposed to be mere geometrical lines, the value of this integral will be in general infinite; if, however, s_1 and s_2 are made of wires of small but definite cross-sections, the finite limit, as s_2 is moved into close contact with s_1, of the flux of magnetic induction caused by the unit current in s_2 through a diaphragm bounded by s_1 is practically the flux through the diaphragm due to the unit current in s_1.

The number of lines (unit tubes) of magnetic induction which thread a simple fine wire circuit made of non-magnetic material, which carries a steady current of unit strength when there are no other currents and no permanent magnets in its neighborhood, is very nearly equal to what is called the *coefficient of self-induction* or the *self-inductance* of the simple circuit, under the circumstances. The numerical value of this coefficient, which we shall soon be able to define more accurately, depends very much upon the nature of the media about the circuit.

81. Coefficients of Induction. If two fine wire closed circuits of non-magnetic material, exactly alike in size and shape, and carrying in the same direction steady currents of intensity C' and C'' respectively, are placed as nearly as possible in coincidence, the coefficient of mutual induction of the two is practically the same as the coefficient of self-induction (L) of either, and the work required to separate the two circuits to an infinite distance from each other is $C'C''L$. If, then, a fine wire closed circuit which carries a steady current C be supposed made up of infinitely slender closed circuit filaments lying freely in contact, it is easy to get an expression for the work that must be done in removing these filaments one after another out of the field. If at some stage in the process the remaining filaments carry altogether the current $C - C''$, the work required to remove another filament carrying the current dC'' would be $(C - C'')\,dC'' \cdot L$, and this integrated with respect to C'' between 0 and C yields $\frac{1}{2}\,C^2L$, which is an expression for the intrinsic energy of the original collection of filaments. Again, if a current C be set up and kept steady in any closed circuit in a medium of any kind which contains no permanent magnets and no other currents, the medium becomes polarized by induction and is a field of force. The electrokinetic energy is equal to the volume integral taken over all space of $\mu C^2R^2/8\,\pi$, where R is the intensity of the field due to a unit current in the conductor. It is easy to see that this reduces in the case of a linear circuit to $\frac{1}{2}\,C^2$ times what we have called the coefficient of self-induction of the circuit, and we are led to define the coefficient of self-induction of a circuit, made up of conductors of any form surrounded by media the susceptibilities of which are independent at every point of the intensity of the force at the point, as twice the energy in the magnetic field when the circuit carries a current of one electromagnetic unit and there are no other currents and no permanent magnets in the neighborhood.

If, for instance, a uniformly distributed current C be carried lengthwise in a homogeneous, infinitely long cylinder of revolution, of radius a, and be brought back in a thin cylindrical shell of inside radius b and outside radius c, coaxial with the cylinder, there is no field without the shell; the intensity of the field is $2\,Cr/a^2$ within the cylinder, $2\,C/r$ between the cylinder and the shell, and $2\,C(c^2 - r^2)/r\,(c^2 - b^2)$ in the shell itself. Neglecting the space occupied by the thin shell, which would contribute little to the result, the whole energy in the field per unit length of the cylinder is

$$\frac{\mu_1}{8\,\pi}\,\frac{4\,C^2}{a^4}\int_0^a 2\,\pi r^3 dr + \frac{\mu_2}{8\,\pi}\,4\,C^2\int_a^b 2\,\pi/r\cdot dr.$$

If the medium between the shell and the cylinder has the uniform inductivity μ_2, this energy is $\frac{1}{4}\,\mu_1 C^2 + \mu_2 C^2 \log b/a$. The coefficient of self-induction of the circuit per unit length is, therefore, when the shell is thin, $\frac{1}{2}\,\mu_1 + 2\,\mu_2 \log b/a$.

The coefficient of self-induction, in electromagnetic absolute c.g.s. units, of a circular ring of circumference l, made of non-magnetic wire of radius r and surrounded by air, is, according to Kirchhoff, $2\,l\,[\log(l/r) - 1.508]$, and that of a square circuit of perimeter l, made of similar wire, $2\,l\,[\log(l/r) - 1.910]$. Regarding the coefficient of self-induction from the point of view of the energy in the field, it is possible to prove that the coefficient of a part of a circuit consisting of a straight wire of length l and radius r is approximately $2\,l\,[\log(2\,l/r) + \frac{1}{4}\,\mu - 1]$, where μ is the magnetic permeability of the wire. For additional examples, the reader is referred to Winkelmann's *Handbuch der Physik*, Vol. III, Maxwell's *Treatise on Electricity and Magnetism*, Vol. II, and to Gray's *Absolute Measurements in Electricity and Magnetism*, Vol. II.

If X_1, Y_1, Z_1 are the components of the electromagnetic field which a unit current flowing in a given circuit s_1 of self-inductance L_1 would cause if the surrounding space contained no other currents and no permanent magnets, and if this space is already the seat of a magnetic field X, Y, Z, caused either by currents or by permanent magnets, or by both, then if a steady current C_1 be set up and maintained in s_1, the electrokinetic energy is

$$\frac{1}{8\,\pi}\iiint_{\infty} \mu\,\{(C_1X_1 + X)^2 + (C_1Y_1 + Y)^2 + (C_1Z_1 + Z)^2\}\,d\tau.$$

The integrand can be split up into three terms,

$$\mu C_1^2[X_1^2 + Y_1^2 + Z_1^2], \quad \mu[X^2 + Y^2 + Z^2],$$

and
$$2\mu C_1[X_1X + Y_1Y + Z_1Z],$$

and the corresponding integrals represent respectively $\frac{1}{2} C_1^2L_1$, the energy of the original field, and that part of the electrokinetic energy due to the introduction of the current into the field. If the external field is due to a steady current C_2 in a second circuit s_2 of self-inductance L_2, the second integral is $\frac{1}{2} C_2^2L_2$, and if the third be written C_1C_2M, the whole energy becomes $\frac{1}{2} C_1^2L_1 + MC_1C_2 + \frac{1}{2} C_2^2L_2$. The quantity M, which in the case where s_1 and s_2 are linear is the coefficient of mutual induction of the two circuits, serves to define this coefficient in the case of circuits which are not linear, surrounded by media which have susceptibilities independent of the strength of the field.

If n circuits which have self-inductances $L_1, L_2, L_3, \cdots$ and carry currents $C_1, C_2, C_3, \cdots$ exist together in a soft medium, and if the mutual inductance of the pth and kth circuits is M_{pk}, the electrokinetic energy T is equal to

$$\tfrac{1}{2}(L_1C_1^2 + L_2C_2^2 + L_3C_3^2 + \cdots + L_nC_n^2)$$
$$+ M_{12}C_1C_2 + M_{13}C_1C_3 + \cdots + M_{1n}C_1C_n + M_{23}C_2C_3 + \cdots,$$

where the values of the inductances depend upon the configuration of the system. If this configuration is determined by a number of generalized coördinates $q_1, q_2, q_3, \cdots$, the electrodynamic force, in the Lagrangian sense, which tends to increase any one of these coördinates (leaving the rest unchanged) is the partial derivative of T with respect to this coördinate. If every circuit is rigid, the L's are constant during any change of configuration.

82. Maxwell's Current Equations. Various Current Systems. We may infer from experiment that if a unit magnetic pole be moved about a simple closed path in any steady electromagnetic field, whether the medium in which the part lies is homogeneous or not, the work done on it by the field is equal to $4\pi C$, where C is the whole current which passes in positive direction through any surface or diaphragm which caps the path. If u, v, w are the components of the current intensity, the flux through the cap may be written in the form

$$\int\!\!\int [u \cdot \cos(x, n) + v \cdot \cos(y, n) + w \cdot \cos(z, n)]\, dS,$$

and if L, M, N are the components of the magnetic force H, the line integral of H taken around the path is equal, according to Stokes's Theorem, to

$$\int\!\!\int [(D_yN - D_zM) \cdot \cos(x, n) + (D_zL - D_xN) \cdot \cos(y, n) \\ + (D_xM - D_yL) \cdot \cos(z, n)]\, dS.$$

It follows that the integral

$$\int\!\!\int [(4\pi u - D_yN + D_zM) \cos(x, n) \\ + (4\pi v - D_zL + D_xN) \cos(y, n) \\ + (4\pi w - D_xM + D_yL) \cos(z, n)]\, dS$$

must vanish whatever the shape of the cap and, therefore, that at every point

$$4\pi u = D_yN - D_zM,$$
$$4\pi v = D_zL - D_xN,$$
$$4\pi w = D_xM - D_yL. \qquad [205]$$

These are Maxwell's Current Equations, which can be stated in the single vector equation

$$4\pi q = \text{curl of } H.$$

This has been called by Heaviside "the first circuital equation" of the electromagnetic field. It states that 4π times the resolved part of the current intensity at any point within a conductor, in any direction, is equal to the resolved part in the given direction of the curl of the magnetic force. The equation holds even in a non-homogeneous medium.

Maxwell's Equations, with the characteristic volume and boundary differential equations which the magnetic induction, as we have seen, must always satisfy, completely determine a steady magnetic field in given media, when the current q is known.

In any *homogeneous* soft medium the magnetic force H is solenoidal, and we may infer from the work of Section 69 that it has a vector potential function Q equal to Pot q. We have, therefore, $H = \text{curl } Q$, $4\pi q = \text{curl } H$, and, if the components of H and Q are L, M, N and Q_x, Q_y, Q_z respectively,

$$Q_x = \iiint_\infty \frac{u_1 d\tau_1}{r}, \quad Q_y = \iiint_\infty \frac{v_1 d\tau_1}{r}, \quad Q_z = \iiint_\infty \frac{w_1 d\tau_1}{r};$$

$$L = D_y Q_z - D_z Q_y, \quad M = D_z Q_x - D_x Q_z, \quad N = D_x Q_y - D_y Q_x.$$

When in a steady field H is known, Maxwell's Equations, or their equivalent, give the current vector q directly. If, for example, the magnetic force is zero everywhere without an infinitely long cylindrical surface S of any shape, while within S the field has the uniform strength N, and is directed parallel to the generating lines of the surface, q is zero within and without S. To show that S itself is a current surface, let KK' be a portion of a generating line drawn in the direction of the field within, and let AB and CD be lines each of length l parallel and close to KK', one within and the other without S, drawn so that AC and BD are normal to the surface. The line integral of the magnetic force taken around the perimeter of the rectangle $ACDB$ is numerically equal to lN, so that, by Stokes's Theorem, the surface integral of the normal upward component of the curl taken over the area of the rectangle is lN, and this is equal to 4π times the steady flow of electricity through the rectangle. There is, therefore, a uniform flow of electricity in S perpendicular to its generating lines equal to $N/4\pi$ per unit length of the surface.

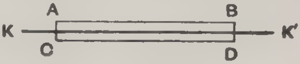

FIG. 74.

This is practically the case of an electromagnetic solenoid, that is, an infinitely long cylindrical surface wound uniformly (and as nearly perpendicularly to the axis of the cylinder as possible) with turns of fine wire. If there are n turns on each centimetre of length of the cylinder (Fig. 75) and if each turn carries a steady current C, $N/4\pi = nC$, or $N = 4\pi nC$.

This result is independent of the magnetic inductivity of the homogeneous soft medium within the cylinder. The induction in the medium is $4\pi n\mu C$, and the intensity of its polarization (magnetization) is $4\pi nkC$ or $nC(\mu - 1)$. The coefficient of self-induction per unit length of the solenoid is $4\pi n^2\mu A$, where A is the area of the cross-section of the cylinder.

If a part of the space within the solenoid be taken up with a homogeneous soft medium of permeability μ_1, and the rest by an infinitely long cylinder of another homogeneous soft medium of permeability μ_2, the lines of which are parallel to those of the surface upon which the wire is wound, the lines of force are unchanged in form, the induction in the first medium is $4\pi n\mu_1 C$ and in the other $4\pi n\mu_2 C$. If A_1 and A_2 represent the portions of the cross-section A of the solenoid occupied by the two media, the self-inductance of the solenoid per unit length is $4\pi n^2(\mu_1 A_1 + \mu_2 A_2)$.

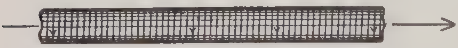

FIG. 75.

The coefficient of mutual induction of two infinitely long solenoids S_1, S_2, one of which has n_1 turns and the other n_2 turns per unit of its length, is zero, unless one, say S_2, is within the other. In this case the coefficient has the value $4\pi n_1 n_2 A_2$ per unit length of the two, where A_2 is the area of the cross-section of S_2.

If two infinitely long, cylindrical surfaces, whatever their shapes may be, have parallel generating lines, and if one of these surfaces is within the other, the space between the surfaces will be a uniform field of magnetic force of strength N, directed parallel to the generating lines, and the regions without the outer surface and within the inner one will be fields of no force, if a uniform current of strength $N/4\pi$ per unit length flows in each surface perpendicular to the generating lines and if the directions of flow around the two surfaces are opposed.

If the two infinite parallel planes $x=a$, $x=b$ carry uniform currents parallel to the y axis, of strength $N/4\pi$ per unit width of the planes parallel to the z axis, and if the directions of the two currents are opposite, the region between the planes is a uniform field of force of strength N parallel to the z axis. There is no force without the space included between the planes. The current in each plane evidently gives rise to a uniform field of intensity $\frac{1}{2}N$ on both sides of the plane.

If a ring surface be formed by revolving about the z axis an area in the xz plane, and if electricity be supposed to flow symmetrically on the surface, in closed paths which lie in planes through the z axis, and coincide with perimeters of cross-sections of the ring formed by such planes, the field has the same intensity at all points of any one of the family (f) of circumferences, the centres of which lie in the z axis and have that line for their common axis. If, using columnar coördinates (r, θ, z), we denote the force components at any point, taken in the directions in which these coördinates increase most rapidly, by R, Θ, Z, these components are independent of θ. Since the amount of work which would be done on a magnetic pole if it were carried around any closed path without the surface, — whether or not it linked with the surface, — or around any *evanescible* path wholly within the surface, would be zero, we are led to guess that the field outside the ring is everywhere zero, and that the lines of force within the ring are circumferences of the f family. If a unit magnetic pole were carried about one of these circumferences of radius r, the work done on it would be $\pm 2\,\pi r\Theta$, and this is equal in absolute value to $4\,\pi E$, where E is the whole amount of electricity which flows about the ring per second. We learn, therefore, that $\Theta = 2\,E/r$. We may now prove that if there is no field without the ring surface, and if the only component within is $\Theta = 2\,E/r$, the currents which give rise to the field must be those assumed above. The components of the field within the ring, taken parallel to rectangular axes, are $-2\,Ey/r^2$, $2\,Ex/r^2$, 0, so that the force is lamellar within and without the surface of the ring. To find what currents flow in the surface itself, we may use a circumference s of radius r, in which a plane perpendicular to the z axis intersects the surface, draw two arcs parallel and very close to s, one on either side, so that one is within the surface and the other without it, and complete a narrow closed contour by drawing two radii (Fig. 76) which make with each other any convenient angle ϕ. Only one side of the contour yields any contribution to the line integral $(2\,E\phi)$, taken about it, of the tangential component of the force. This integral measures the work done on a unit magnetic pole carried around the contour, and is equal to $4\,\pi$ times the strength of the current across the portion of s, of length $r\phi$, which the contour encloses. If the whole flux across s is F, the flux across this arc is $\phi F/2\,\pi$, and we have the equation, $2\,E\phi = 4\,\pi\phi F/2\,\pi$, or $F = E$.

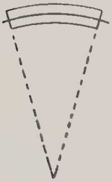

Fig. 76.

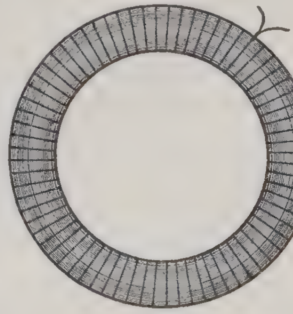

Fig. 77.

The case here considered is approximately that of a ring of revolution wound uniformly with fine wire (Fig. 77) in turns which lie nearly in radial planes through the axis of the ring. If there are n turns on the ring, and if each turn carries the steady current C, $E = nC$, and the force within the ring is $2nC/r$, whatever the inductivity of the homogeneous soft medium within the ring. The induction in the medium is $2\mu nC/r$, and the intensity of its magnetization is $2knC/r$. It is to be noted that the reasoning here employed might be applied unchanged if the inductivity of the medium were a function of r and z, but not a function of θ; this would be the case, for instance, if into air space within the ring were introduced a soft iron ring coaxial with this space.

A slender magnetic filament within the ring surface, of length l and cross-section S, carries $2\mu ES/r$, or $4\pi nC/(l/\mu S)$ lines of induction. The line integral of the magnetic force taken along a magnetic filament in a soft medium is sometimes called the *magnetomotive force* in it, and the ratio of this quantity to the flow of induction in it the *reluctance* of the filament. In the case before us $4\pi nC$ is the magnetomotive force, and $l/\mu S$ the reluctance. This last expression bears a close resemblance to the formula for the electric resistance of a wire of length l, cross-section S, and specific conductivity μ. The reciprocal of the reluctance of a magnetic filament in a soft medium is sometimes called its *permeance*.

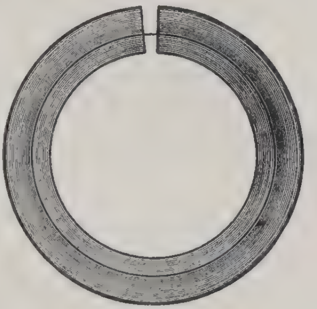

FIG. 78.

If wire were wound part way around a soft iron ring, in the manner described above, most of the lines of induction would still be confined to the iron, though a few would emerge into the air at the ends of the coil.

If a radial gap be cut in a soft iron ring completely wound with wire, the field is no longer symmetrical about the axis of z, and the character of the problem is changed. The line integral of the tangential force taken around a circumference inside the ring, of radius r (Fig. 78), with centre on the z axis and plane perpendicular to that axis, is still $4\pi E$ or $4\pi n C$, but the portion of the path in air now contributes far more than its due proportion to the result, and the path in the iron much less than before. We know that at any surface of discontinuity in the inductivity of a soft medium the normal component of the induction is continuous, so that if the normal component of the force in the iron, just where the path is about to emerge into the gap, is H, that in the air near by is μH, where μ is the inductivity of the iron. Although the lines of force within the iron are no longer exactly circular, they are nearly so, and the line integral of the force about the circumference just mentioned is very approximately $Hr(2\pi - \phi) + \mu H r \phi$, or $4\pi n C$; and $H = 2nC / \{r[1 + \phi(\mu - 1)/2\pi]\}$, where ϕ is the angle subtended by the gap at the axis of the ring. If, in the case of the core used, $\mu = 1201$, and if only one per cent of the ring be cut away, the induction in the iron will be reduced to about one-thirteenth of its old value; the reluctance of the path will be increased thirteenfold.

If the lines of force in a steady electromagnetic field are all circles with centres on the z axis and planes perpendicular to this axis, and if the intensity of the force in a direction linked right-handedly with the z axis is $f\sqrt{x^2 + y^2}$ or $f(r)$, it is evident that $L = -y \cdot f(r)/r$, $M = x \cdot f(r)/r$, $N = 0$, so that $u = 0$, $v = 0$, $4\pi w = D_x M - D_y L = f'(r) + f(r)/r = D_r[r \cdot f(r)]/r$. According to this, if in any portion of the field $f(r) = 0$, in that portion w is 0; if $f(r) = c$, $w = c/4\pi r$; if $f(r) = c/r$, $w = 0$; and if $f(r) = cr$, $w = c/2\pi$. If, on the other hand, while u and v are zero, w is given as a function of r, $f(r)$ can be obtained from the equation $f(r) = \frac{4\pi}{r}\int rw\,dr$; when, therefore, w is equal to the constant w_0, $f(r) = 2\pi w_0 r + d/r$. If the cylindrical surface $r = b$ separates two regions in a field of this kind where the laws of force intensity $f_1(r)$, $f_2(r)$ in the inner and outer of these regions are different, and if $f_2(b) - f_1(b) = k$, it is easy to see, with the help of Stokes's

Theorem, that the surface $r = b$ is itself a current surface in which there is a total flux parallel to the z axis across any right section of $\frac{1}{2}kb$.

Up to this time we have considered only media which have inductivities independent of the magnetizing force. The permeabilities of the so-called magnetic metals do not in general satisfy this condition, and we may note in passing that some of our definitions have to be restated when there are masses of such media near a circuit.

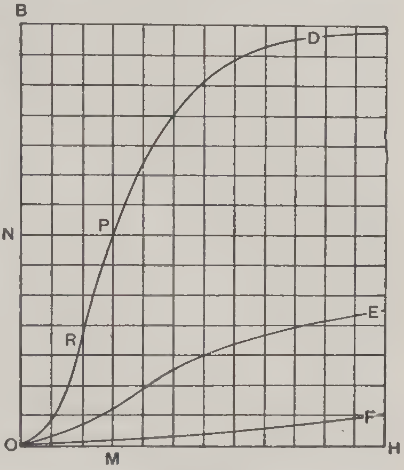

Fig. 79.

If fine wire, carrying a steady current, be wound uniformly upon a cylindrical rod of soft iron, the length of which is at least 400 times its diameter, the induction at the middle of the rod is sensibly the same as if the rod were infinitely long, and if this induction be measured (in a manner to be described in a later section), the permeability of the iron may be determined. Curves D, E, F of Fig. 79, in which the abscissas represent the magnetizing force H in units, and the ordinates the corresponding induction $B \equiv \mu H$ in thousands of units, show the results of experiments upon specimens of very soft malleable iron, soft cast iron, and very hard steel, respectively. It is evident that so far from being constant, the permeability and the susceptibility of each of these specimens increase to a maximum at a value of H corresponding to a point where a tangent from the origin touches the curve, and then decrease. In the case of the curve D, for instance, the permeability, corresponding to a value OM of the magnetizing force, is MP / OM, that is, the slope of the straight line OP joining the origin with the point on the curve which has OM for its abscissa.

When the conductors which make up a simple linear circuit which carries a steady current C, and the soft media about it, have inductivities independent of the magnetizing force, and there are no other currents and no permanent magnets in the field, the coefficient of self-induction of the circuit may be defined indifferently as the ratio of the total induction through the circuit to C or as twice the ratio of the integral of $\mu H^2/8\pi$, taken over the field of the current, to C^2. In this case the magnetizing curves of all the substances in the field are straight lines, and these definitions lead to the same value whatever C is. If the magnetizing curve of any medium in the field were, like that of soft iron, not straight, the definitions would not agree, and each would yield different values for different values of C.

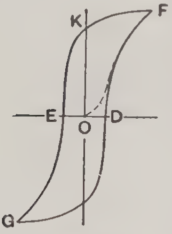

FIG. 80.

Mechanically soft iron or steel cannot be regarded as magnetically soft, for if a piece of either of these metals be magnetized by induction, this magnetization does not wholly disappear when the magnetizing force is removed. If the magnetizing force be made to change continuously from a given negative value to an equal positive value and back several times, the induction goes through a cycle which may be represented graphically by a curve somewhat like that shown in Fig. 80, in which the abscissas represent magnetizing forces, and the ordinates the corresponding values of the induction. Such diagrams make plain the fact that the induction in a piece of soft iron or steel is not a definite function of the magnetizing force, and that the energy in the medium, as defined by the volume integral of $1/8\pi$ times the product of the numerical values of the induction and the magnetizing force, may have, for the same force, very different values, depending on the previous history of the metal. When the induction has passed through such a cycle as that indicated in Fig. 80, the energy in the field returns to its old value, but it is easy to prove that an amount of work represented by $1/4\pi$ times the area of the cycle per unit volume of the substance had to be done on the metal during the cycle, and that this appeared as heat. The reader will find the subject which has been just touched upon here admirably treated under the head of "Hysteresis" in Ewing's *Magnetic Induction in Iron and Other Metals*, and in Fleming's *The Alternate Current Transformer.*

IV. CURRENT INDUCTION.

83. Electromagnetic Induction. If one of two circuits (s_1, s_2), so situated that their coefficient of mutual induction is not zero, contains a galvanic cell and a key, and the other (s_2), which is permanently closed, a galvanoscope, a momentary current appears in s_2 when the key is depressed so that a current circulates in s_1; and another momentary current, opposed in direction to the first, runs through s_2 when the key is opened again. A current in either s_1 or s_2 gives rise to a magnetic field and causes lines of magnetic induction to thread s_2: the direction of the transient current in s_2 in each of the cases mentioned is such that the lines which it threads through s_2 oppose the sudden change in the flux of induction through s_2 which the change in the current in s_1 tends to cause. Thus, if the relative position of the two circuits and the direction of the current in s_1 are correctly indicated in Fig. 81, the transient induced current in s_2 will flow from B to A when the key is depressed and from A to B when the key is again opened. In general, if a rigid, closed circuit s is in a magnetic field caused either by permanent magnets or by electric currents in neighboring circuits,

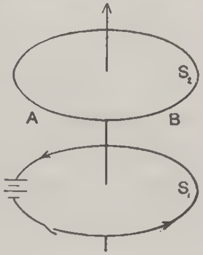

FIG. 81.

or by both together, and if the positive flux Q of magnetic induction through any cap or diaphragm bounded by s be changed in amount, either by moving s or by changing the field in any way, a temporary current is induced in s in a direction which tends to oppose the change in Q. The phenomenon is quantitatively explained, when s is unchanged in form, by assuming that, superposed upon such electromotive forces as the circuit may already contain, a temporary electromotive force numerically equal to the time rate of change of Q is induced in s in the proper direction.

Transient currents are usually induced also in any circuit in a magnetic field when the circuit is deformed or extended in any way. These currents, like those already considered, are mathematically accounted for on the supposition that there is induced in every circuit element ds, which moves in a magnetic field so as to cut across the lines of induction during the motion, an electromotive force numerically equal to the time rate at which the element cuts these lines. This electromotive force is directed from the feet to the head of an observer who, lying in the element and looking along the lines of force, sees these lines move past him from right to left. The induced current at any instant in either direction around the circuit is equal to the ratio of the algebraic sum of the electromotive forces induced at that instant, in that direction, to the whole resistance of the circuit. If in Fig. 82 OC represents the direction of a circuit element at the point O, OM the direction in which the element is moved, and OF the direction of the whole field at O, the induced electromotive force will have the direction OC, not CO. The direction of the current, induced by the motion, in the direction OM, of a circuit element at O in a magnetic field which has there the direction OF, may be found by choosing that direction, OC, in the element which will cause the three directions OC, OM, OF to be related like those of the x, y, z axes of a Cartesian system. It is to be noticed

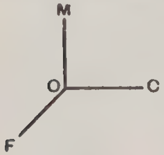

FIG. 82.

that the direction of the current induced in an element is such that the mechanical action of the field upon the element carrying this current alone would hinder the motion; a circuit element carrying a current in the direction *OC* in a field having the direction *OF* in Fig. 83 would be urged in a direction *ON* perpendicular to the plane of *FOC* and would move in the direction *MO*, if free to do so, rather than in the direction *OM*. The reader will do well to compare, in this connection, Figs. 68 and 82.

If (a, b, c), (α, β, γ) are the components of two vectors, l and λ, the vector which has the components $(c\beta - b\gamma, a\gamma - c\alpha, b\alpha - a\beta)$ is sometimes called their *vector product* and the quantity $-(a\alpha + b\beta + c\gamma)$ their *scalar product*. The vector product of l and λ has a direction perpendicular to the plane of these vectors: its tensor is the product of their tensors and the sine of the angle between their directions. The electromotive force induced in or impressed upon an element ds of a linear conductor moving in a magnetic field is evidently equal to the product of ds and the component in its direction of the vector product of the induction and the velocity of the element.

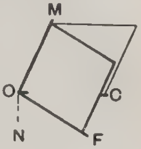

Fig. 83.

If (B_x, B_y, B_z) are the components of the induction, (ξ, η, ζ) those of the velocity of the element relative to the field, and if the induction does not change with the time, the absolute value of the electromotive force induced in the element is

$$[(B_z \cdot \eta - B_y \cdot \zeta)\cos(x, s) + (B_x \cdot \zeta - B_z \cdot \xi)\cos(y, s) + (B_y \cdot \xi - B_x \cdot \eta)\cos(z, s)]\,ds. \quad [206]$$

The whole electromotive force induced in the conductor is the integral of this expression: if the conductor is not closed this electromotive force gives rise to a statical distribution of electricity on the ends of the conductor, and hence to a difference of electrostatic potential which tends to destroy itself by causing a current in the conductor in the direction opposite to the impressed electromotive force.

If the induction (B_x, B_y, B_z) of the magnetic field in the neighborhood of a fixed linear circuit changes with the time, the induced or impressed electromotive force e in the circuit is equal to the negative of the surface integral, taken over any cap S bounded by the circuit, of

$$[D_tB_x \cdot \cos(x, n) + D_tB_y \cdot \cos(y, n) + D_tB_z \cdot \cos(z, n)].$$

If, then, a vector can be found of which the vector

$$(D_tB_x, D_tB_y, D_tB_z)$$

is the curl, then the line integral, taken around the circuit, of the tangential component of this new vector — increased, if we please, by any lamellar vector — will be equal in absolute value to the induced electromotive force. If (F_x, F_y, F_z) is any vector potential (Section 69) of the induction, (D_tF_x, D_tF_y, D_tF_z) is a vector potential function of (D_tB_x, D_tB_y, D_tB_z), and if $-\psi$ is the scalar potential function of any lamellar vector, the integral, taken around the circuit in positive direction, of

$$-[(D_tF_x + D_x\psi)\cos(x, s) + (D_tF_y + D_y\psi)\cos(y, s) + (D_tF_z + D_z\psi)\cos(z, s)] \quad [207]$$

will be equal to e. This value of the whole electromotive force induced in the circuit will be obtained if we assume that every circuit element ds is the seat of an electromotive force equal to the product of ds and the tangential component of the vector $-[D_tF_x + D_x\psi, D_tF_y + D_y\psi, D_tF_z + D_z\psi]$.

If a closed linear circuit s in a magnetic field be deformed or moved according to any law so that during the time dt the coördinates (x, y, z) of the beginning of an element ds receive increments $(\delta x, \delta y, \delta y)$ which are analytic functions of x, y, z, and dt, and if, during the interval dt, the scalar point functions which represent the magnitudes of the components of the magnetic induction in the field change from the values

B_x, B_y, B_z to the values B_x', B_y', B_z', the flux of induction through the circuit has been increased by the amount

$$d\Phi = \iint [B_x' \cdot \cos(x, n) + B_y' \cdot \cos(y, n) + B_z' \cdot \cos(z, n)]\, dS'$$
$$- \iint [B_x \cdot \cos(x, n) + B_y \cdot \cos(y, n) + B_z \cdot \cos(z, n)]\, dS,$$

where S' and S are any surfaces which cap the circuit in its final and initial positions respectively. In moving, the circuit traces out a narrow surface S'', each element ds of the circuit generating the surface element dS'', and we may take for the cap S' the surface made up of S and S''. We have therefore $d\Phi$

$$= dt \iint [D_t B_x \cdot \cos(x,n) + D_t B_y \cdot \cos(y,n) + D_t B_z \cdot \cos(z,n)]\, dS$$
$$+ \iint [B_x' \cdot \cos(x, n) + B_y' \cdot \cos(y, n) + B_z' \cdot \cos(z, n)]\, dS''.$$

In the second integral, $\cos(z, n) \cdot dS''$ measures the area of the projection of dS'' on the xy plane and is, therefore, equal to $\pm(\delta x \cdot dy - \delta y \cdot dx)$ plus terms of higher order; the sign being positive, if the direction in which ds moves, the positive direction of the element, and that of the normal to dS'' are arranged like the x, y, z axes of a Cartesian system. We may substitute in the integrand B_x, B_y, B_z, $\xi \cdot dt$, $\eta \cdot dt$, $\zeta \cdot dt$ for B_x', B_y', B_z', δx, δy, δz, without changing the value of the integral, and then write

$$D_t\Phi = \int [P \cdot \cos(x, s) + Q \cdot \cos(y, s) + R \cdot \cos(z, s)]\, ds,$$

where
$$P \equiv - D_t F_x - D_x \psi + B_z \cdot \eta - B_y \cdot \zeta,$$
$$Q \equiv - D_t F_y - D_y \psi + B_x \cdot \zeta - B_z \cdot \xi, \qquad [208]$$
$$R \equiv - D_t F_z - D_z \psi + B_y \cdot \xi - B_x \cdot \eta.$$

P, Q, R are said to be the components of the *induced electromotive force* at the element ds.

We may note in passing that we cannot generally assume that the motion of the electricity in a circuit which is the seat of an induced current is governed by a potential function due to an electrostatic distribution on the surfaces of the conductors, or elsewhere. If a magnet, the axis of which coincides with the axis of a plane circular ring of wire, be made to approach or to recede from the plane of the ring, a transient current is induced in the wire, but no imaginable electrostatic distribution would furnish the multiple-valued potential function needed to account for the current.

If a circuit at a distance from other circuits and permanent magnets carries a changing current C, the ratio of the numerical value of the intensity of the electromotive force induced by the change of the current in the circuit to D_tC is sometimes used as a definition of the self-inductance of the circuit. The mutual inductance of two circuits may be defined in a similar manner. It is evident that all the definitions of self and mutual inductance which we have mentioned are equivalent when all the media in the neighborhood of the circuits concerned have susceptibilities independent of the intensity of the field. The definitions of this section are often used when there are masses of soft iron or other magnetic metals near the circuits, or when the circuits themselves are made of soft iron conductors.

If a number of circuits $s_1, s_2, \cdots s_n$, carrying currents $C_1, C_2, \cdots C_n$, have self-inductances $L_1, L_2, \cdots L_n$, and if the mutual inductance of s_k and s_l is M_{kl}, the total electrokinetic energy T is of the form

$$\begin{aligned}
&\tfrac{1}{2} L_1 C_1^2 + \tfrac{1}{2} L_2 C_2^2 + \cdots \tfrac{1}{2} L_n C_n^2 \\
&+ M_{12} C_1 C_2 + M_{13} C_1 C_3 + \cdots + M_{23} C_2 C_3 + M_{24} C_2 C_4 + \cdots \\
&+ M_{34} C_3 C_4 + \cdots + M_{n-1,n}\, C_{n-1}\, C_n,
\end{aligned}$$

where the L's and the M's are independent of the C's and are to be considered as functions of a set of geometrical coördinates equal in number to the degrees of freedom of the system. If p_k denotes the electrokinetic momentum of s_k, that is, the partial derivative of T with respect to C_k, if r_k represents the resistance of s_k, and E_k the internal electromotive force in this circuit, $\frac{dp_k}{dt}$ measures the intensity of the induced electromotive force, and

$$E_k - \frac{dp_k}{dt} = r_k C_k.$$

If the relative positions of two rigid circuits s_1, s_2, which carry currents C_1, C_2, and are surrounded by a soft medium in which there are no other conductors, be altered by changing under their mutual action the geometrical coördinate q by the amount dq in the time interval dt, leaving the other coördinates which determine the configuration unchanged, the electrokinetic energy $T \equiv \frac{1}{2} L_1 C_1^2 + MC_1 C_2 + \frac{1}{2} L_2 C_2^2$ will receive the increment

$$dT \equiv L_1 C_1 \cdot dC_1 + L_2 C_2 \cdot dC_2 + M(C_2 \cdot dC_1 + C_1 \cdot dC_2) + C_1 C_2 \cdot dM.$$

The electrodynamic force (in the Lagrangian sense) which tends to bring about this change of configuration is the partial derivative of T with respect to q, taken under the assumption that the other coördinates and the currents are constant: the work done during the change by this force is $dW \equiv C_1 C_2 \cdot dM$. Within the circuits we have

$$E_1 C_1 \cdot dt - C_1 \cdot d(L_1 C_1 + MC_2) = C_1^2 r \cdot dt,$$
$$E_2 C_2 \cdot dt - C_2 \cdot d(L_2 C_2 + MC_1) = C_2^2 r_2 \cdot dt,$$

so that the work done against the inductive electromotive forces by the applied electromotive forces (besides the amount $C_1^2 r_1 + C_2^2 r_2$ dissipated in heat) is

$$C_1 \cdot d(L_1 C_1 + MC_2) + C_2 \cdot d(L_2 C_2 + MC_1),$$

or

$$L_1 C_1 \cdot dC_1 + L_2 C_2 \cdot dC_2 + M(C_2 \cdot dC_1 + C_1 \cdot dC_2) + 2\, C_1 C_2 \cdot dM,$$

or

$$dW + dT.$$

If, starting from rest, the circuits come again to rest and the currents regain their steady values before the end of the interval dt, we have

$$dC_1 = 0,\ dC_2 = 0, \text{ and } dW + dT = 2\, C_1 C_2 \cdot dM = 2\, dW.$$

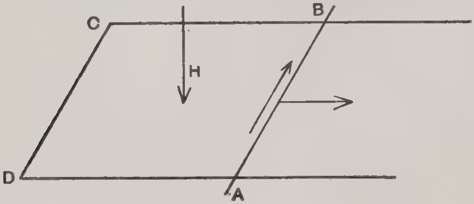

FIG. 84.

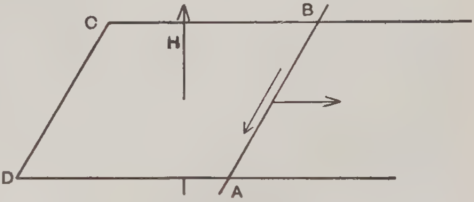

FIG. 85.

The principles just laid down enable us to infer that, if the conductor *AB* of length l in either of the circuits represented by Figs. 84 and 85 be moved parallel to itself along the rails *CB*, *DA*, in the direction indicated by the arrow attached to it, with constant velocity v, and if the field have the direction shown, an electromotive force will be induced in *AB* in the direction pointed out by the arrow by its side. If the component of the total induction normal to the plane of the circuit have the constant value H all along *AB*, and if r be the resistance of the whole circuit *ABCD*, the induced current will be lHv/r in absolute units. The volt, ohm, and ampere are equal respectively to 10^8, 10^9, 10^{-1} times the absolute electromagnetic c.g.s. units of electromotive force, resistance, and current strength; if in this example, therefore, $l = 1$ metre, $v = 1$ metre per second, and $H = 1$, the induced electromotive force will be 10,000 units, or 10^{-4} volts.

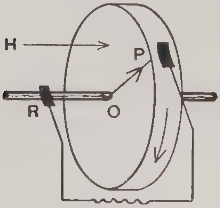

FIG. 86.

If a Faraday's disc (Fig. 86) which has a radius a be rotated in a uniform field, in which the component of the induction normal to the face of the disc is H, with uniform angular velocity ω in the direction indicated, the number of lines of induction cut per second by OP is $\frac{1}{2}a^2H\omega$. If r be the resistance of the circuit, the current in it is $a^2H\omega/2r$ and the disc is a very simple form of constant current generator.

Fig. 87 represents a circuit a part of which consists of a rigid wire free to turn in the air about the axis of a magnet. This wire makes electrical contact, by means of brushes, with the magnet at its mid-section and with a conductor which forms an extension of the axis of the magnet. If the wire be rotated with uniform angular velocity ω, and if m be the strength of one pole of the magnet, the electromotive force induced in the circuit will be $2m\omega$.

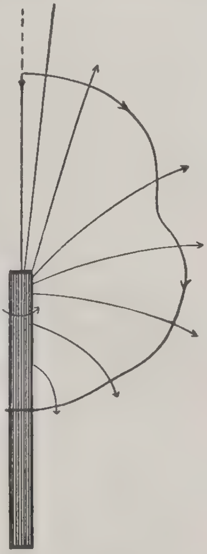

Fig. 87.

If a thin coil (Fig. 88) closely embracing a magnet be suddenly slipped from one position to another, the electromotive force induced in the coil is proportional to the amount of induction which emerges from the surface of the magnet between the two positions.

84. Superficial Induced Currents.

Although a mathematical treatment of the currents induced in a massive conductor of any form, in a magnetic field varying in a given manner, is beyond the scope of this elementary text-book, we may give a very simple proof (taken essentially from Prof. J. J. Thomson's admirable *Elements of Electricity and Magnetism*) of the fact that the currents due to a sudden, finite change in the field lie at the first instant wholly on the surface of the conductor.

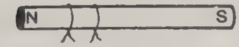

Fig. 88.

Let n linear circuits, the resistances of which are $r_1, r_2, r_3, \cdots$, and the self-inductances $L_1, L_2, L_3, \cdots$, lie near each other in a magnetic field so that the coefficient of mutual induction of the ith and jth circuits is M_{ij}. Let the flux of the external field through the circuits be N_1, N_2, N_3, $\cdots$, and assume that the currents are originally zero and that no one of the circuits contains any battery or other generator. If, then, the external field experiences a finite change during the extremely short time interval τ and thereafter remains constant, the flux through the kth conductor becomes changed from N_k to N_k'. Transient currents, C_1, C_2, C_3, $\cdots$, flow through the circuits and at the end of the time τ attain the values C_1', C_2', C_3', $\cdots$. During the given interval we have in the first conductor, which will serve as a general example,

$$L_1 \cdot D_t C_1 + \sum (M_{1k} \cdot D_t C_k) + D_t N_1 + r_1 C_1 = 0,$$

and if this be integrated with respect to the time between 0 and τ, the last term of the result will be less than $r_1 C_1' \tau$, which is negligible, so that the result may be written in the form $L_1 \cdot C_1' + \Sigma (M_{1k} \cdot C_k') + N_1' = N_1$. The second member represents the whole induction flux through the first circuit before the change and the first member the whole flux at the end of the time τ, so that the currents generated by the sudden change in the field are such as to keep unchanged the whole flux.

Imagine a compact mass of metal divided into such circuits as we have just considered and it will be evident that the flux through every circuit in the metal is the same just after the sudden change in the field as it was before. The work done in carrying a magnetic pole about any closed path in the metal is unaltered by the change: it is zero before the change and zero after. No such path can enclose any current filament and, therefore, all the induced currents are initially on the surface, though afterwards transient currents are excited within the metal. It is easy to infer from this that, if the external magnetic field is a very rapidly alternating one, the induced currents never penetrate very far into the mass of the conductor. For references to the literature of this important subject, the student may consult Winkelmann's

Handbuch der Physik, Vol. III, p. 403. Various problems are discussed at length in J. J. Thomson's *Recent Researches in Electricity and Magnetism.* We shall confine our attention in the three sections which follow to circuits made up of long slender conductors like wires.

85. Variable Currents in Single Circuits. When a simple inductive circuit of resistance r, containing a constant electromotive force E, is suddenly closed, the current in the circuit grows gradually in strength and in a short time practically attains a maximum value $C_0 \equiv E/r$, after which it remains constant. While the current is increasing in intensity, the electromagnetic energy in the surrounding medium — if there are no permanent magnets and no other currents in the neighborhood — increases also from zero to $\frac{1}{2} LC_0^2$, and electrostatic charges are established which account for the electrostatic potential differences in the conductors which make up the circuit. After the current has attained the value C_0 the energy (C_0E watts or $C_0E \cdot 10^7$ ergs per second) given up to the circuit by the generator in it is used in heating the conductors in the circuit, and $EC_0 = C_0^2r$. Before the current C has become steady CE is only a fraction of C_0E, and the rate C^2r, at which energy is used in heating the circuit, is a still smaller fraction of C_0^2r; hence $CE - C^2r$ is positive, and in the time interval dt the energy $(CE - C^2r)\,dt$ joules is used partly in increasing by dw the energy of the electrostatic distribution on the surface of the conductors and elsewhere, and partly in increasing by $d(\frac{1}{2} LC^2)$ or $LC \cdot dC$ the electrokinetic energy in the medium. Unless something in the nature of a condenser is attached to the circuit, dw is usually of no practical importance, and we may write

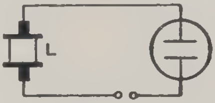

FIG. 89.

$$(CE - C^2r)\,dt = LC \cdot dC, \quad \text{or} \quad Cr = E - L \cdot D_tC,$$

$$\text{or} \quad L \cdot D_tC + Cr = E, \qquad \text{or} \quad C = E/r + A \cdot e^{-rt/L}.$$

It appears from the equation $Cr = E - L \cdot D_t C$ that the counter-electromotive force cannot be greater than E while the current is positive; $D_t C$, therefore, is not greater than E/L and, unless $L = 0$, the current cannot jump at the instant to a finite value. We must assume, then, that $C = 0$ when $t = 0$, so that $C = E(1 - e^{-t/\tau})/r$, where $\tau = L/r$. The quantity $(1 - e^{-t/\tau})$ has the values 0, 0.3935, 0.6321, 0.7769, 0.8647, 0.9179, 0.9502, 0.9817, 0.9933, 0.9975, 0.9991 when the ratio of t to τ has the values 0, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 4.0, 5.0, 6.0, 7.0. The difference $C - C_0$ or $-Ee^{-t/\tau}/r$, which we may call the induced current, has the value $-C_0$ at the beginning and sinks to $1/e$th of this value in τ seconds, which is sometimes called the *relaxation time* of the circuit. The induced electromotive force has the value $-Ee^{-t/\tau}$ and becomes insignificant in a short time after the circuit is closed. The integral, with respect to the time between 0 and ∞, of the induced current, is $-EL/r^2$.

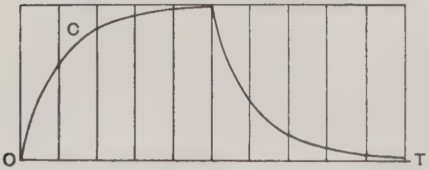

FIG. 90.

If, now, the electromotive force in the circuit be suddenly changed to E', we have at any time t seconds after the change $E'C \cdot dt = C^2 r \cdot dt + LC \cdot dC$ or $C = E'/r + (E - E')e^{-t/\tau}/r$. The induced current is now the second term in this expression for C and the induced electromotive force is never larger than $E - E'$. The quantity $e^{-t/\tau}$ has the values 1, 0.6065, 0.3679, 0.2231, 0.1353, 0.0821, 0.0498, 0.0183, 0.0067, 0.0025, 0.0009 when the ratio of t to τ has the values 0, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 4.0, 5.0, 6.0, 7.0. It is to be noted that if L is expressed in terms of the practical unit (the *henry*) of self-induction, which is equal to 10^9 absolute units, r must be measured in terms of the ohm, which is equal to 10^9 absolute units of resistance. The relaxation time of a circuit, which is sometimes called also its *time constant*, is usually a fraction of a second. The ordinates of the curve in Fig. 90 represent the strength of the current in the circuit just described, on the assumption that the electromotive force is kept constant for 5 τ seconds after the circuit is closed and is then suddenly annihilated.

If, starting with no current in the circuit, the electromotive force have the constant value E_0 for the time interval a, then the value zero during the interval b, then the value E_0 again during an interval a, then the value zero during an interval b, and so on, and if we denote $e^{-a/\tau}$ and $e^{-b/\tau}$ by α and β, the current at the end of the nth period of interruption, $n(a+b)$ seconds from the beginning, will be

$$E_0\beta(1-\alpha)(1+\alpha\beta+\alpha^2\beta^2+\cdots+\alpha^{n-1}\beta^{n-1})/r,$$

and the limit of this, as n increases, is

$$C_0 = E_0\beta(1-\alpha)/r(1-\alpha\beta).$$

Starting with this value C_0, the current during the next period a, while the electromotive force is equal to E_0, would be $E_0(1-e^{-t/\tau})/r + C_0e^{-t/\tau}$, and during the next interval b, when the electromotive force is zero,

$$E_0(1-\alpha)e^{-t/\tau}/r + C_0\alpha e^{-t/\tau}.$$

At the end of this interval the current is again C_0 and the state is final.

If E in the equation $L\cdot D_tC + r\cdot C = E$ is a given function of the time, $L\cdot C = e^{-t/\tau}(A + \int e^{t/\tau}\cdot E\cdot dt)$.

If the resistance of an inductive circuit containing a constant electromotive force E and carrying a steady current $C_0 \equiv E/r$ be *suddenly* changed from r to r', we have at any time after the change $EC\cdot dt = C^2r'\cdot dt + LC\cdot dC$, or, if $C' \equiv E/r'$, $C = C' + (C_0 - C')e^{-r't/L}$. The induced electromotive force is now $r'(C_0 - C')e^{-r't/L}$, and if r' is large this may be at first enormous. Although it is very difficult in practice to increase the resistance of a circuit thus instantaneously, the rate of change in r may easily be made very rapid, and the spark which is often visible when a circuit is broken bears witness to the fact that the induced electromotive force is sometimes large.

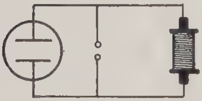

FIG. 91.

If the terminals of a battery of internal resistance r and electromotive force E be connected by a coil of resistance r_1 and self-inductance L_1, in parallel with a non-inductive resistance r_2 (Fig. 91), and if C, C_1, C_2 represent the strengths of the currents in the battery and in the two branches of the external surface respectively,

$$C = C_1 + C_2, \qquad Cr + C_2 r_2 = E, \qquad E - L_1 \cdot D_t C_1 = Cr + C_1 r_1,$$

or $C_1 r + C_2 (r + r_2) = E$, and $C_1 (r + r_1) + C_2 r = E - L_1 \cdot D_t C_1$.

If the value of C_2 from the equation before the last be substituted in the last equation, we get

$$L_1 \cdot D_t C_1 + C_1 R / (r + r_2) = E r_2 / (r + r_2),$$

where $R \equiv r r_1 + r r_2 + r_1 r_2$, so that $C_1 = E r_2 / R + A \cdot e^{-kt}$,

where $k = R / L_1 (r + r_2)$. $C_2 = (E - C_1 r) / (r + r_2)$.

If the main circuit be suddenly closed when $t = 0$, we have

$$C_1 = E r_2 (1 - e^{-kt}) / R.$$

If, after the circuit has been closed for some time and C_1 has attained the value B, the battery be suddenly detached, C^1 and C_2 become suddenly equal numerically,

$$L_1 \cdot D_t C_1 + C_1 (r_1 + r_2) = 0,$$

and $$C_1 = B e^{-mt}, \text{ where } m = (r_1 + r_2) / L_1.$$

If the poles of a constant battery of resistance b are connected by two coils in parallel (Fig. 92) which have resistances r_1, r_2 and self-inductances L_1, L_2, we have

$$L_1 \cdot D_t C_1 + (b + r_1)\, C_1 + b \cdot C_2 = E,$$

$$b \cdot C_1 + L_2 \cdot D_t C_2 + (b + r_2)\, C_2 = E,$$

or

$$(L_1 \cdot D_t + b + r_1)\, C_1 + b \cdot C_2 = E,$$

$$b \cdot C_1 + (L_2 \cdot D_t + b + r_2)\, C_2 = E.$$

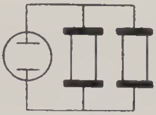

FIG. 92.

If we perform the operations $(L_2 \cdot D_t + b + r_2)$ and b upon the two equations respectively, and subtract one result from the other, we shall get the equation

$$L_1 \cdot L_2 \cdot D_t^2 C_1 + [L_1(b + r_2) + L_2(b + r_1)]\, D_t C_1 + (br_1 + br_2 + r_1 r_2)\, C_1 = r_2 E\,;$$

whence $C_1 = r_2 E / (br_1 + br_2 + r_1 r_2) + A \cdot e^{\lambda t} + B \cdot e^{\mu t}$,

where λ and μ are the roots of the quadratic

$$L_1 \cdot L_2 x^2 + [L_1(b + r_2) + L_2(b + r_1)]\, x + (br_1 + br_2 + r_1 r_2) = 0.$$

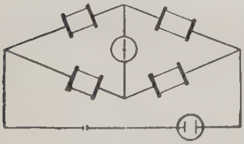

FIG. 93.

Fig. 93 represents a Wheatstone's Net which has self-inductance in all members except that which contains the cell. Using, as far as it goes, the notation of Section 73, let us call the coefficients of self-induction of the branches which have the resistances p, q, r, s, g; L_p, L_q, L_r, L_s, L_g respectively. Let $ps = qr$, so that, when the current has become steady, there is no flow through g, while the current

$$P_0 \equiv C(q + s)/(p + q + r + s)$$

flows through p and r, and the current

$$Q_0 \equiv C(p + r)/(p + q + r + s)$$

through q and s. If, now, the branch b be suddenly broken, transient currents C_p, C_q, C_r, C_s, C_g, which have the initial values P_0, Q_0, P_0, Q_0, zero respectively, and the final value zero, will flow through the members of the rest of the net. Kirchhoff's Laws give at every instant

$$p \cdot C_p + L_p \cdot D_t C_p + g \cdot C_g + L_g \cdot D_t C_g - q \cdot C_q - L_q \cdot D_t C_q = 0,$$
$$r \cdot C_r + L_r \cdot D_t C_r - s \cdot C_s - L_s \cdot D_t C_s - g \cdot C_g - L_g \cdot D_t C_g = 0,$$
$$C_p - C_g - C_r = 0.$$

If we multiply each of these equations by dt, integrate between $t = 0$ and $t = \infty$, and write

$$P \equiv \int_0^\infty C_p \cdot dt = -\int_0^\infty C_q \cdot dt, \quad R \equiv \int_0^\infty C_r \cdot dt = -\int_0^\infty C_s \cdot dt,$$
$$G \equiv \int_0^\infty C_g \cdot dt,$$

we shall get the equations

$$(p+q)P + gG = L_p \cdot P_0 - L_q \cdot Q_0,$$
$$(r+s)R - gG = L_r \cdot P_0 - L_s Q_0,$$
$$P - R - G = 0.$$

Whence,

$$G = \frac{C[(r+s)L_p - (p+q)L_r](q+s) + C[(p+q)L_s - (r+s)L_q](p+r)}{(p+q+r+s)[g(p+q+r+s) + (p+q)(r+s)]},$$

or, since $ps = qr$,

$$G = \frac{Cps(L_p/p - L_r/r + L_s/s - L_q/q)}{g(p+q+r+s) + (p+q)(r+s)}.$$

If L_q and L_s are both zero (Fig. 94), it is possible to choose r and p subject to the condition $ps = qr$, so that there shall be no transient current through g, and in this case $L_p/L_r = p/r$. If L_q, L_r, L_s are all zero, and if the steady current C and the quantity G be measured, L_p can

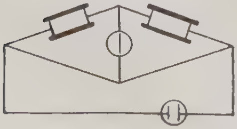

FIG. 94.

be found. This method of determining coefficients of self-induction is described at length by Lord Rayleigh in the *Philosophical Transactions* for 1882.

If at the time t the positive plate of a condenser of capacity K, which is being charged by a battery of constant electromotive force E (Fig. 95), has a charge Q; if r is the resistance of the "circuit," L its coefficient of self-induction, and $C \equiv D_tQ$, the charging current, we have

$$E - Q/K - L \cdot D_tC = rC \quad \text{or} \quad L \cdot D_t^2Q + r \cdot D_tQ + Q/K = E.$$

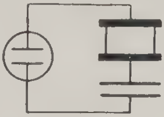

FIG. 95.

The general solution of this equation for Q is the sum of any special solution (for instance, KE) and the general solution of the equation formed by equating the first member to zero. If, therefore, $\lambda_1 \equiv -r/2L + R$ and $\lambda_2 \equiv -r/2L - R$, where $R^2 \equiv r^2/4L^2 - 1/KL$, the solution required is of the form $KE + ae^{\lambda_1 t} + be^{\lambda_2 t}$, where a and b are constants to be determined from the initial conditions. If the absolute value of the quantity under the radical sign in the expressions for λ_1 and λ_2 — taken positive, whatever its real sign may be — is m^2, the value of the radical will be m or mi according as r^2 is greater or less than $4L/K$. If at the time zero, when $Q = Q_0$, the circuit be suddenly closed,

$$Q = KE + (Q_0 - KE)(\lambda_2 \cdot e^{\lambda_1 t} - \lambda_1 \cdot e^{\lambda_2 t})/(\lambda_2 - \lambda_1).$$

The current has the value $\lambda_1\lambda_2(Q_0 - KE)(e^{\lambda_1 t} - e^{\lambda_2 t})/(\lambda_2 - \lambda_1)$, and if λ_1 and λ_2 are real, it has the same sign for all values of t. If, however, λ_1 and λ_2 are imaginary, the expression given above for Q may be more conveniently written in the form $KE + (Q_0 - KE)e^{-rt/2L}(\cos mt + r/2Lm \cdot \sin mt)$, and the sign of the second term is alternately positive for π/m seconds and negative for π/m seconds, so that the current is sometimes positive and sometimes negative. The curves in Fig. 96 exhibit Q and C in terms of t in a case where $r^2 > 4\,L/K$, $Q_0 = 0$, and the condenser is being charged; the curves in Fig. 97 correspond to a case where $E = 0$ and the condenser is discharging itself through the circuit. In each case the absolute value of the current starts at zero, attains a maximum, and then decreases. Fig. 98 shows Q in terms of t when $E = 0$ and the condenser is discharging itself through a circuit such

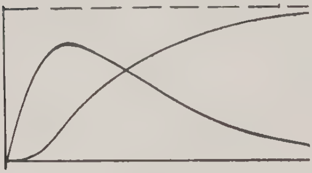

Fig. 96.

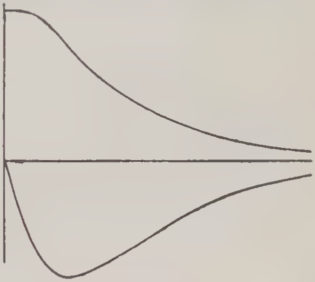

Fig. 97.

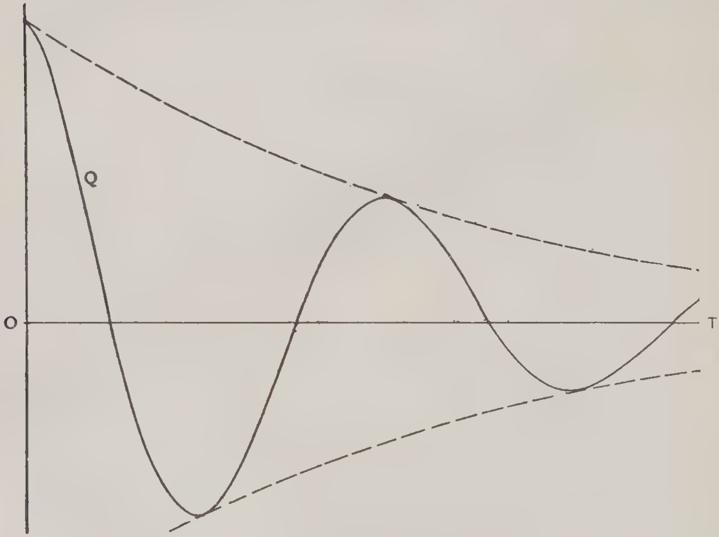

Fig. 98.

that $r^2 < 4\,L/K$; the curve, the ordinates of which are EK minus the ordinates of this curve, shows Q at any time while the condenser is being charged by the battery. The shape of the curve may be seen by looking at Fig. 98 through the back of the leaf and upside down.

If we differentiate the equation $E - Q/K - L \cdot D_t C = rC$ with respect to t, we get $L \cdot D_t^2 C + r \cdot D_t C + C/K = D_t E = 0$, and we might determine C directly from this last equation.

If a condenser of capacity K, originally charged to potential Q_0/K, be discharged through a circuit (Fig. 99) which consists of a non-inductive resistance r_1 and an inductive resistance r_2, arranged in multiple arc, and if the currents at the time t through the branches of the external circuit be C_1 and C_2 respectively, $C_1 + C_2 = -D_t Q$. If we take into account the induced electromotive force, we may apply Kirchhoff's Laws directly to this circuit and learn that $Q/K - C_1 r_1 = 0$, and that

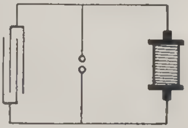

FIG. 99.

$$Q/K - L_2 \cdot D_t C_2 - C_2 r_2,$$

$$\text{or } Q/K + L_2 \cdot D_t(D_t Q + C_1) + r_2(D_t Q + C_1) = 0.$$

If the values $C_1 = Q/Kr_1$, $D_t C_1 = D_t Q/Kr_1$, obtained from the first of these equations, be substituted in the last one, it becomes $L_2 \cdot D_t^2 Q + D_t Q\,(L_2/Kr_1 + r_2) + Q\,(r_1 + r_2)/Kr_1 = 0$, and the solution of this is of the form $ae^{\lambda t} + be^{\mu t}$, where λ and μ are the roots of the equation

$$L_2 x^2 + (L_2/Kr_1 + r_2)\,x + (r_1 + r_2)/Kr_1 = 0.$$

After a and b have been determined in accordance with the given conditions, C_1 and C_2 can be found directly. The equation $C_2 r_2 = Q/K - L_2 \cdot D_t C_2$ shows that, if C_2 is positive, $D_t C_2$ cannot be greater than Q/KL_2, and that C_2 cannot jump suddenly from zero to a finite value as soon as the condenser circuit is closed; the initial value of C_2 is therefore zero, while that of C_1 is Q_0 / Kr_1, and under the conditions of this problem

$$Q = Q_0[(\mu Kr_1 + 1) e^{\lambda t} - (\lambda Kr_1 + 1) e^{\mu t}] / Kr_1 (\mu - \lambda),$$

$$C_1 = Q_0[(\mu Kr_1 + 1) e^{\lambda t} - (\lambda Kr_1 + 1) e^{\mu t}] / K^2 r_1^2 (\mu - \lambda),$$

$$C_2 = - Q_0 (\mu Kr_1 + 1)(\lambda Kr_1 + 1)(e^{\lambda t} - e^{\mu t}) / K^2 r_1^2 (\mu - \lambda).$$

If λ and μ are real, C_1 decreases from the value Q_0 / Kr_1 to zero, C_2 starts at zero, increases (accompanied by a self-induced counter-electromotive force $L_2 \cdot D_t C_2$, so that $C_2 r_2 < C_1 r_1$) until it attains a maximum at the time

$$t = (\log \lambda - \log \mu) / (\mu - \lambda),$$

at which time $D_t C_2$ vanishes and $C_1 r_1 = C_2 r_2$, and then continually decreases, accompanied by a self-induced positive electromotive force, so that $C_2 r_2 > C_1 r_1$. If we integrate C_1 with respect to the time from $t = 0$ to $t = \infty$, and remember that $\lambda + \mu = -(L_2 + Kr_1 r_2) / L_2 K r_1$, and that

$$\lambda\mu = (r_1 + r_2) / L_2 K r_1,$$

we shall obtain the whole flow $Q_0 r_2 / (r_1 + r_2)$ through r_1. This is the same (whether or not λ and μ are real) as if r_2 had no self-inductance; but if the circuit be broken before the discharge is complete, a greater portion of the electricity will have gone through r_1 than would be the case if L_2 were zero.

If the condenser connections have a considerable resistance b, the differential equation becomes

$$KL_2(b + r_1) D_t^2 Q + [L_2 + K(br_1 + br_2 + r_1 r_2)] D_t Q + (r_1 + r_2) Q = 0.$$

If the resistance of a circuit made up of a generator of electromotive force $E \equiv f(t)$, a condenser of capacity k and necessary leads, is r, if its self-inductance is L, and if we denote D_tE by E', we have $L \cdot D_t^2C + r \cdot D_tC + C/K = E'$. If $R \equiv \sqrt{r^2K^2 - 4\,LK}$, and

$$\alpha \equiv (rK - R)/2\,LK, \quad \beta \equiv (rK + R)/2\,LK,$$

the general solution of this equation is

$$C = (K/R)\left(e^{\beta t}\int e^{-\beta t} \cdot E' \cdot dt - e^{\alpha t}\int e^{-\alpha t} \cdot E' \cdot dt\right) + A \cdot e^{-\alpha t} + B \cdot e^{-\beta t}.$$

If the poles of a battery of constant electromotive force E and internal resistance b are connected by a coil of resistance r_1 and self-inductance L_1, in parallel with a condenser of capacity K (Fig. 100), we have

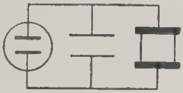

Fig. 100.

$$L_1 \cdot D_tC_1 + (b + r_1)\,C_1 + b \cdot C_2 = E,$$

and $$E - Q/K = bC_1 + (b + r_2)\,C_2,$$

or $$b \cdot D_tC_1 + (b + r_2)\,D_tC_2 + C_2/K = 0.$$

If we perform on the first and last of these equations the operations $[(b + r_2)\,D_t + 1/K]$ and b respectively, and subtract one result from the other, we shall learn that

$$KL_1(b + r_2)\,D_t^2C_1 + [L_1 + K(br_1 + br_2 + r_1r_2)]\,D_tC_1 + (b + r_1)\,C_1 = E,$$

and that C_1 is the sum of $E/(b + r_1)$ and the general solution of the equation formed by putting the first number equal to zero.

If the arms p and r in the Wheatstone Net contain condensers of capacity K_p, K_r respectively, the steady current through q and s will be $C = E/(b + q + s)$, and the charges of the condensers will be CqK_p and CsK_r. If, now, the battery circuit be suddenly broken, transient currents will appear in the remaining members of the net and the condensers will be discharged. The whole flow through p will be $-CqK_p$, and that through r will be $-CsK_r$. At any instant during the discharge $C_p = C_r + C_g$, and if we multiply this equation by dt and integrate between 0 and ∞, it will appear that the whole flow through g is $C(sK_r - qK_p)$. This will be zero if $q/s = K_r/K_p$.

86. Alternate Currents in Single Circuits. In many practical applications of electricity it is necessary to deal with inductive circuits which contain periodic electromotive forces. In the simplest case the electromotive force is harmonic of the form $E_m \cdot \sin(pt - \alpha)$, or the form $E_m \cdot \cos(pt - \alpha)$; the *amplitude* is then E_m; the *period*, $T \equiv 2\pi/p$; the *frequency*, $n \equiv p/2\pi$; and the *phase lag*, α.

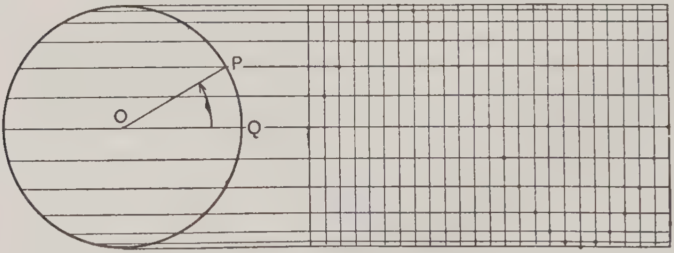

FIG. 101.

Two harmonic electromotive forces, of the same period, $A \cdot \sin(pt - \alpha)$, $B \cdot \sin(pt - \beta)$, which conspire in a simple circuit, are equivalent to a single simple harmonic electromotive force $C \cdot \sin(pt - \gamma)$, where $C^2 = A^2 + B^2 + 2AB \cdot \cos(\alpha - \beta)$ and $\tan\gamma = (A \sin\alpha + B \sin\beta)/(A \cos\alpha + B \cos\beta)$. If a parallelogram be constructed with adjacent sides equivalent on any scale to A and B, and with the included angle equal to $(\alpha - \beta)$, a diagonal of the parallelogram will represent C, and the angles which this diagonal makes with adjacent sides of the parallelogram will be equal to $(\alpha - \gamma)$ and $(\gamma - \beta)$ respectively.

If, starting at the time $t = 0$ from the position P_0, a point P be made to move uniformly with angular velocity p in counter-clockwise direction around the circumference of a circle with centre O and radius E_m, if Q be a fixed point in the plane of the circumference such that $P_0OQ = \alpha$, and if y be any straight line in the plane perpendicular to OQ, the projections of OP on OQ and on y will be equal, at any time t, to

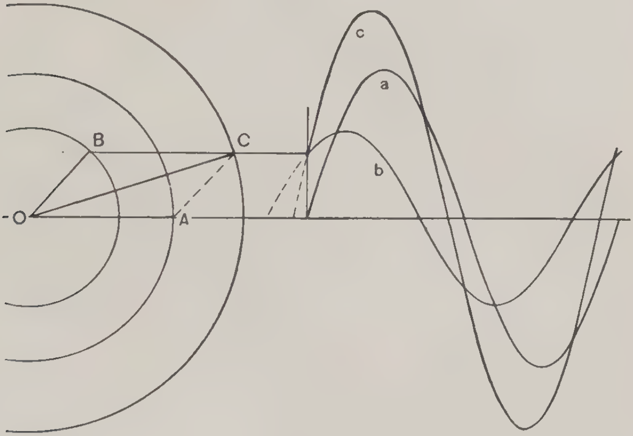

FIG. 102.

$E_m \cdot \cos(pt - \alpha)$ and $E_m \cdot \sin(pt - \alpha)$ respectively. If OQ be used as an axis of real quantities, QOP will represent the argument, and the length of OP the modulus of the complex quantity $E_m \cdot e^{(pt-\alpha)i}$; the real part and the real factor of the imaginary part of this quantity will be represented by the projections of OP on OQ and on y.

If while P is moving in the circumference, y moves parallel to itself away from O with constant velocity a / T or $ap / 2\pi$, the projection of P upon y will trace out a sinusoid (Fig. 101) the length of the base of which is a. We shall frequently find it convenient to imagine diagrams as generated in this way. If in Fig. 102 the lines OA, OB, OC revolve uniformly in the plane of the diagram about O with angular velocity p, if the angle $AOB = \beta$, and if the lengths OA, OB are equal to the amplitudes of two simple harmonic quantities $a \equiv a_m \cdot \sin pt$, $b \equiv b_m \cdot \sin(pt + \beta)$, the projections of A, B, and C on y give the curves a, b, and c; every ordinate of the sinusoid c is the sum of the corresponding ordinates of the sinusoids a and b.

If in Fig. 103 the independent lines OA, OB, OC revolve about O in the plane of the diagram with the same constant angular velocity p, the lengths of the projections of these lines upon any fixed line (z) in the plane will represent harmonic quantities of the same frequency $(p/2\pi)$, but with phase differences equal to the angles between the lines projected. The sum of these harmonic quantities may be represented by the projection upon z of QD, which is equivalent to the geometric sum of OA, OB, OC, if QD revolve about Q with angular velocity p, starting to move at the same time with the original lines.

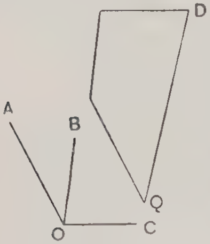

Fig. 103.

If a circuit s which has a resistance r and a self-inductance L contains an electromotive force $E_m \cdot \cos pt$, we have $L \cdot D_t C + rC = E_m \cdot \cos pt$, if the capacity of the circuit is negligible. The complete solution of this equation is the sum of any special solution and the complete solution, $Ae^{-rt/L}$, of the equation formed by writing the first member equal to zero. To find the special solution needed, we may consider first the equation $L \cdot D_t C + rC = E_m (\cos pt + i \sin pt) = E_m \cdot e^{pti}$, which is in some respects simpler; if any solution of this new equation has the form $u + vi$, u is a solution of the given equation. Since the first member of the new equation is linear in terms of C and $D_t C$, and since $D_t e^{pti} = pie^{pti}$, it is clear that a special solution of this equation, of the form Be^{pti}, must exist.

Substituting this form in the equation, to determine B, we learn that the solution is

$$E_m \cdot e^{pti} / (r + Lpi), \text{ or } E_m (r - Lpi) e^{pti} / (r^2 + L^2p^2),$$

of which the real part is $E_m (r \cos pt + Lp \sin pt) / (r^2 + L^2p^2)$, or

$$\frac{E_m}{\sqrt{r^2 + L^2p^2}} \cos (pt - \alpha), \text{ where } \tan \alpha = Lp / r.$$

The current in s is, therefore,

$$Ae^{-rt/L} + E_m \cdot \cos (pt - \alpha) / \sqrt{r^2 + L^2p^2},$$

but after a comparatively short time the first term becomes negligible, and then the current becomes harmonic with the same period, $2\pi/p$, and the same frequency, $p/2\pi$, as the electromotive force, but with a retardation in phase of α. The amplitude is $E_m / \sqrt{r^2 + L^2p^2}$. The radical $\sqrt{r^2 + L^2p^2} \equiv Z$ is called the *impedance* of the circuit and $Lp \equiv x$ its *reactance*, or *inductive resistance*, under the given circumstances; the self-induction of the circuit reduces the amplitude of the current in the ratio of r to Z. The relation between the electromotive force and the current strength may be represented by corresponding ordinates of two curves like those shown in Fig. 104.

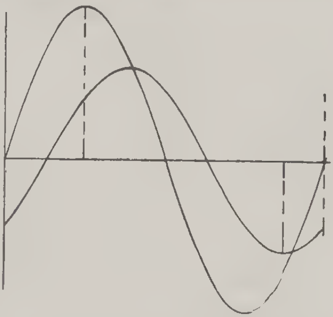

FIG. 104.

The counter-electromotive force of self-induction, sometimes called the *back electromotive force* of self-induction, is equal to $- L \cdot D_t C$, or $\frac{Lp \cdot E_m}{Z} \cdot \cos (pt - \alpha - \frac{1}{2}\pi)$; it lags 90° behind the current in phase. The electromotive force necessary to overcome self-induction is the opposite of this; it has the same numerical value, but its phase is 90° in advance of that of the current. If we denote the amplitude, E_m/Z, of the current by C_m, the amplitude of the electromotive force necessary to overcome self-induction will be LpC_m. Cr is called the *apparent* electromotive force or the *instantaneous energy component* of the electromotive force; its amplitude is rC_m. The amplitude of the *applied* electromotive force $E_m \cos pt$ is ZC_m.

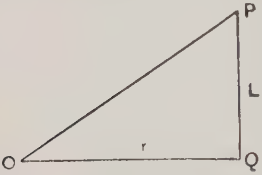

Fig. 105.

If a right triangle be drawn (Fig. 105) the legs of which represent r and Lp on any scale, the hypotenuse will represent Z on the same scale and the angle between the r and Z sides will be α; this triangle is the *triangle of resistances.* A triangle OPQ (Fig. 106) similar to this, the sides of which are equal to rC_m, LpC_m, and ZC_m, may be called the *triangle of electromotive forces.* If the figure $OQPR$ be made to rotate positively about O with constant angular velocity p, the projections at any time of OQ, OP, OR upon any line in the plane of the diagram parallel to the original position of OP will give the electromotive forces at that instant.

The *activity* or the energy spent in the circuit, during any time interval, at the expense of the generator is the time integral, taken over that interval, of $EC \equiv E_m^2 \cdot \cos pt \cdot \cos(pt - \alpha) \cdot / Z$. The mean value of the activity for any number of whole periods is $E_m^2 r / 2(r^2 + L^2p^2)$, and this is the same as if a steady current of intensity $E_m / \sqrt{2(r^2 + L^2p^2)}$ had passed through the circuit during the interval; for this reason $E_m / \sqrt{2(r^2 + L^2p^2)}$ is said to be the *virtual* or *effective* current. The mean values for any number of whole periods of the current and of the square of the current are zero and $E_m^2 / 2Z^2$; the effective current is, therefore, the square root of the mean square of the current, and this is sometimes called the *quadratic mean.* The *effective applied*

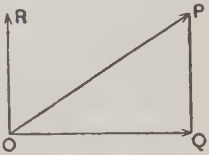

Fig. 106.

electromotive force is $E_m / \sqrt{2}$, and the *effective apparent* electromotive force is $E_m r / \sqrt{2} \cdot Z$. The apparent electromotive force would yield the current C if applied to a circuit of ohmic resistance r and inductive resistance zero. The activity, or "power in the circuit," is equal for any number of whole periods to the product of the effective current and the effective apparent electromotive force. For this reason the effective apparent electromotive force is frequently called the *effective energy component of the electromotive force.* The first term $E_m{}^2 r \cdot \cos^2(pt - a)/Z^2$ of the second member of the equation $EC = C^2 r + LC \cdot D_t C$ shows the rate at which heat is being dissipated in the circuit; the second term,

$$- E_m{}^2 Lp \cdot \sin(pt - a) \cdot \cos(pt - a)/Z^2,$$

the rate at which power is used in increasing the energy of the electromagnetic field. It is evident that the average value of this last quantity for any number of whole periods is zero. The effective impressed electromotive force is often called simply "the electromotive force." Such voltmeters and ammeters as are commonly used in alternating circuits usually indicate *effective* electromotive forces and currents; their readings must be multiplied by $\sqrt{2}$ to obtain the maximum values of these quantities.

It is often convenient, as Prof. C. A. Adams has pointed out, to regard the values, at any instant of the impressed electromotive force and of the current, as the projections, on the real axis, of the radii vectores which join the origin to the two points on the complex plane which represent at that instant the quantities $E_m \cdot e^{pti}$, $E_m \cdot e^{pti} / (r + Lpi)$. This last expression is the simple solution already found for the differential equation $L \cdot D_t C + r \cdot C = E_m \cdot e^{pti}$.

If in the problem just considered we reckon the time from an epoch $\frac{1}{4} T$ earlier, we shall have

$$E = E_m \cdot \sin pt, \quad C = E_m \cdot \sin(pt - a) / Z;$$

these quantities may be regarded as the projections on the axis of imaginaries of the moduli of $E_m \cdot e^{pti}$ and $E_m \cdot e^{pti} / (r + Lpi)$. The quantity $(r + Lpi)$ has been called the *complex impedance*, but some writers give this name to $r - Lpi$.

If a linear plane circuit of area A, resistance r, and self-inductance L, in a uniform magnetic field in air of intensity H, be made to rotate about an axis perpendicular to the lines of the field with angular velocity p, and if at the time $t = 0$ the plane of the circuit is parallel to the field, the flux of the field through the coil at the time t is $AH \sin pt$, and the current C satisfies the equation $L \cdot D_t C + Cr = - pAH \cos pt$, so that after a few seconds $C = - HAp \cdot \cos (pt - \alpha) / Z$. The whole flow of electricity through the circuit during a positive half revolution is $2\,HA / Z$. The mechanical action between the circuit and the field is equivalent to a couple the moment of which is C times the rate of change with respect to pt of the flux $AH \sin pt$ through the coil. This moment is $CHA \cos pt$, or $- H^2A^2p \cdot \cos pt \cdot (\cos pt - \alpha) / Z$, its average value is $- H^2A^2pr / 2Z^2$, and the work done against it in a single revolution is $H^2A^2r\pi p / Z^2$. External work must be done to turn the coil against the resistance of this couple, and the equivalent of this work is all used in heating the circuit. If the rate of rotation is so rapid that the ratio of r to Lp is small, α is nearly equal to $\frac{1}{2}\pi$, and C is nearly equal to $- HA \sin pt / L$; CL is the flux through the circuit of the lines of its own field, $HA \sin pt$ is the corresponding flux of the lines of the external field, and in this case the sum of the two is nearly zero.

If two points A and B in an inductive circuit be joined by an additional (non-inductive) conductor which carries an electromotive force of such a value at every instant and so directed that no current ever passes through the extra conductor, the electromotive force may be taken as a measure of the difference of potential between A and B. If between A and B in a simple circuit which carries the current $C_m \cdot \sin(pt - \alpha)$ there is an ohmic resistance r and a self-inductance L, the difference of potential between the points is evidently

$$\sqrt{r^2 + L^2p^2} \cdot C_m \cdot \sin(pt - \delta),$$

where $\tan\delta = (r \cdot \sin\alpha - Lp\cos\alpha) / (r \cdot \cos\alpha + Lp \cdot \sin\alpha)$. If the terminals of an alternating current voltmeter were attached to A and B, the instrument would measure $C_m Z / \sqrt{2}$.

If a circuit which carries a current $C_m \cdot \cos pt$ contains three coils in series which have resistances r_1, r_2, r_3 and inductances L_1, L_2, L_3, we may lay off on a horizontal line in succession (Fig. 107) the lengths $OA = r_1 C_m$, $AB = r_2 C_m$, $BQ = r_3 C_m$. Erect at Q a vertical line and lay off on it the lengths

$$QD = L_1 p C_m,\ DF = L_2 p C_m,\ FP = L_3 p C_m.$$

Then OP will represent the amplitude of the difference of potential between O and P, and QOP will be the angle of advance of its phase over that of the current. The lines a, b, c represent similarly the amplitudes of the differences of potential of the ends of the separate coils, and the angles which these lines make with the horizontal the phase differences between these potential differences and the current. Starting at the time zero, let the triangle OQP revolve about O, in the plane of the diagram, with constant angular velocity p, and let the initial position of OQ be denoted by OQ_0. Let the points of intersection of the lines a, b and b, c be denoted by G and H, and the projections of A, B, Q, D, F, P, G, H upon OQ_0 by corresponding accented letters; then the lengths at any time of the lines OG', $G'H'$, $H'P'$ represent the instantaneous values of the "electromotive forces" applied to the several coils, and the lengths of OA', $A'B'$, $B'Q'$ the corresponding apparent electromotive forces.

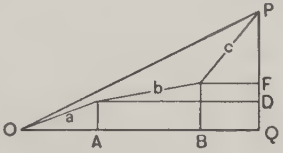

FIG. 107.

If the terminals of a generator of electromotive force $E_m \cdot \cos pt$ and internal resistance r are connected by two conductors in parallel (Fig. **108**) of resistance r_1, r_2 and of self-inductance L_1, L_2 respectively,

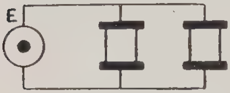

FIG. 108.

$$L_1 \cdot D_t C_1 + (r + r_1)\, C_1 + rC_2 = E_m \cdot \cos pt,$$

$$L_2 \cdot D_t C_2 + rC_1 + (r + r_2)\, C_2 = E_m \cdot \cos pt.$$

If r is negligible,

$$C_1 = E_m \cdot \cos(pt - a_1) / \sqrt{r_1^2 + L_1^2 p^2} = A \cdot \cos(pt - a_1),$$

and

$$C_2 = E_m \cdot \cos(pt - a_2) / \sqrt{r_2^2 + L_2^2 p^2} = B \cdot \cos(pt - a_2),$$

where $\tan a_1 = L_1 p / r_1$, $\tan a_2 = L_2 p / r_2$.

$$C_1 + C_2 = C_m \cdot \cos(pt - a),$$

where $C_m^2 = A^2 + B^2 + 2\, AB \cdot \cos(a_1 - a_2)$

and $\tan a = (A \cdot \sin a_1 + B \cdot \sin a_2) / (A \cdot \cos a_1 + B \cdot \cos a_2)$.

If in Fig. 109, $OP = E_m$ and $QOP = a_1$, $OQ = Ar_1$, $QP = AL_1 p$, and A can be represented by a length laid off from O on OQ. A similar construction, represented by the dotted lines, may be made for B. The diagonal OR of the parallelogram, two sides of which are the lines which represent A and B, represents C_m. If OR cuts the semicircumference in G, OG represents the product of C_m and the resistance of the divided circuit.

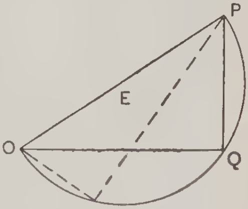

FIG. 109.

If a simple harmonic difference of potential $E_m \cdot \cos pt$ be applied to two points A and B which are connected by n simple conductors of resistances r_1, r_2, r_3, $\cdots$, self-inductances L_1, L_2, L_3, $\cdots$, and impedances Z_1, Z_2, Z_3, $\cdots$; and if the sum of the n fractions of the form r_k / Z_k^2 be denoted by F and that of the fractions of the form pL_k / Z_k^2 by G, the n conductors are equivalent to a single conductor of resistance $R \equiv F/(F^2+G^2)$ and of reactance $X \equiv G/(F^2+G^2)$. The sum of the currents in all the conductors is $E_m \cdot \cos(pt-\alpha)/\sqrt{R^2+X^2}$, where $\tan\alpha = X/R$.

If a non-inductive circuit of resistance r containing a condenser of capacity k and a generator of electromotive force $E \equiv E_m \cdot \sin pt$ be suddenly closed at the time $t=0$, and if Q is the charge on the positive plate of the condenser at the time t, $E - Q/k = rC$, or, since $C = D_tQ$,

$$r \cdot D_tC + C/k = pE_m \cdot \cos pt.$$

From this it follows that

$$C = Ae^{-t/rk} + E_m \cdot \sin(pt+\beta)/\sqrt{r^2+m^2},$$

and $\quad Q = B - Arke^{-t/rk} + E_m \cdot \sin(pt+\beta-\tfrac{1}{2}\pi)/p\sqrt{r^2+m^2}$

where $m = 1/pk$, and $\tan\beta = 1/rpk$.

The exponential terms soon become negligible, and if we assume that Q is zero at the outset, we shall have eventually

$$C = E_m \cdot \sin(pt+\beta)/\sqrt{r^2+m^2},$$

or $pE_mk \cdot \cos(pt-\delta)/\sqrt{1+k^2p^2r^2}$, where $\tan\delta = prk$;

$$Q = E_m \cdot \sin(pt+\beta-\tfrac{1}{2}\pi)/p\sqrt{r^2+m^2}.$$

Here the phase of the current is in advance of that of the applied electromotive force E by the angle β and in advance of Q by 90°. The electromotive forces of the condenser and generator conspire in direction when pt lies between $n\pi$ and $n\pi+\alpha$, where n is any integer and $\alpha = 90° - \beta$; these electromotive forces are opposed when pt lies between $n\pi+\alpha$ and $(n+1)\pi$. The electromotive force (Q/k) necessary to overcome that of the condenser lags behind that of the generator by α. The apparent electromotive force

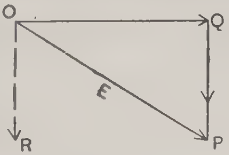

FIG. 110.

is rC. The sum of the squares of the amplitudes (rC_m and mC_m) of rC and Q/k is E_m^2, the square of the amplitude of E; if, therefore, we draw a right triangle of which rC_m and mC_m are legs, the hypotenuse will be equal to E_m, and the angle adjacent to the first leg will be β. If such a triangle OPQ (Fig. 110) be made to rotate in counter-clockwise direction, with constant angular velocity p about O, the projections of OQ, OP, and OR upon any line perpendicular to the initial position of OP will give the apparent electromotive force, the applied electromotive force, and the electromotive force necessary to overcome that of the condenser.

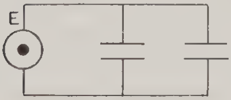

Fig. 111.

If a condenser of capacity k_1 furnished with leads of resistance r_1 be joined in parallel (Fig. 111) with a condenser of capacity k_2 furnished with leads of resistance r_2, and if the compound condenser thus formed be connected up with a generator of internal resistance r and electromotive force $E_m \cdot \sin pt$, we have

$$(r + r_1)D_tC_1 + r \cdot D_tC_2 + C_1/k_1 = pE_m \cdot \cos pt,$$
$$r \cdot D_tC_1 + (r + r_2)D_tC_2 + C_2/k_2 = pE_m \cdot \cos pt.$$

If $r = 0$, we have eventually

$$C_1 = pE_mk_1 \cdot \cos(pt - \alpha_1) / \sqrt{1 + k_1^2p^2r_1^2},$$
$$C_2 = pE_mk_2 \cdot \cos(pt - \alpha_2) / \sqrt{1 + k_2^2p^2r_2^2},$$

where $\tan \alpha_1 = pr_1k_1$, $\tan \alpha_2 = pr_2k_2$.

If a circuit of resistance r contains (Fig. 112) a generator of electromotive force $E_m \cdot \sin pt$, a coil of self-inductance L, and a condenser of capacity k in series, and if Q is the charge on the positive plate of the condenser at the time t, $C = D_tQ$

and $$E_m \cdot \sin pt - Q/k - L\,.\,D_tC = Cr,$$

or $$L \cdot D_t^2C + r \cdot D_tC + C/k = pE_m \cdot \cos pt.$$

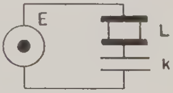

Fig. 112.

The real part of any solution of the equation

$$L \cdot D_t^2 C + r \cdot D_t C + C / k = pE_m e^{pti}$$

(and one evidently exists of the form Be^{pti}) will be a special solution of the equation just formed. It is easy to find B by substituting Be^{pti} in the new equation, and to prove that $E_m \cdot \sin(pt - \alpha) / R$, where $R^2 = r^2 + (pL - 1/kp)^2$ and $\tan \alpha = (p^2kL - 1)/pkr$, is the result required. To obtain the complete solution of the equation for C we should need to add to this special solution the complete solution (found in the last section) of the equation formed by writing the first member equal to zero; this solution is exponential in form, with negative indices increasing in absolute value with the time, so that after a few seconds the current may be represented by the equation $C = E_m \cdot \sin(pt - \alpha) / R$. It is to be noticed that the capacity of the condenser tends to offset in some respects the effect of the self-induction of the coil. Since $R^2 = r^2 + p^2(L - 1/p^2k)^2$ and $\tan \alpha = p(L - 1/p^2k)/r$, it is clear that the current in the circuit is the same as if the condenser were removed and the self-inductance decreased by $1/p^2k$. The maximum current is obtained when both self-inductance and capacity are absent, or when both are present and such that $Lkp^2 = 1$. If $Q = Q_0$ when C has its maximum value, the difference of potential (Q/k) between the plates of the condenser is $Q_0/k - E_m \cdot \cos(pt - \alpha)/pRk$, and if the denominator of the harmonic term is less than unity, this term will have an amplitude greater than that of the impressed force. If we make k infinite in these expressions, they become applicable to the case of a simple inductive circuit containing no condenser. The radical R, which is called the *impedance* of the circuit, becomes $\sqrt{r^2 + L^2p^2}$ when k is infinite.

When an inductive circuit contains a generator of electromotive force $E_m \cdot \sin pt$ and an electrolytic cell with polarizable electrodes, we may assume that when the frequency is fairly large the counter-electromotive force in the cell at any instant is approximately equal to $1/k$ times the quantity of electricity which has passed through the cell in the direction which the current then has, since the last reversal. On this hypothesis the cell acts like a condenser; k depends only upon the electrolyte used and upon the size and material of the electrodes. Experiment shows that if similar platinum electrodes of moderate size be used, the capacity, per square millimetre of the surface of either electrode, will be about 0.049, 0.089, 0.183, 0.049 microfarads, according as the electrolyte is a dilute solution in water of K_2SO_4, KCl, KBr, or KI.

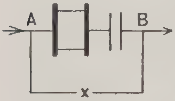

FIG. 113.

If between A and B in a simple circuit (Fig. 113) which carries the current $C \equiv C_m \cdot \sin(pt - a)$ there is a resistance r, a self-inductance L, and a condenser of capacity k in series with the self-inductance, the difference of potential between these two points is $rC + L \cdot D_tC + Q/k$. If $Q = Q_0$ when $C = C_m$,

this is $\quad Q_0/k + C_m \cdot \sqrt{r^2 + (Lp - 1/pk)^2} \cdot \sin(pt - \delta)$,

where $\quad \tan \delta = \dfrac{rp \cdot \sin a + (1/k - Lp^2) \cos a}{rp \cdot \cos a + (Lp^2 - 1/k) \sin a}$.

If the ends of a coil of resistance r_1 and self-inductance L_1, which is joined up with a generator of resistance r and electromotive force $E_m \cdot \sin pt$, be connected by leads of resistance r_2 with the terminals of a condenser of capacity k_2, the coil and the condenser are in parallel (Fig. 114), and

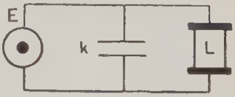

FIG. 114.

$$L_1 \cdot D_tC_1 + (r + r_1)C_1 + rC_2 = E_m \cdot \sin pt,$$

$$r \cdot D_tC_1 + (r + r_2)D_tC_2 + C_2/k_2 = pE_m \cdot \cos pt.$$

If r is negligible,

$$C_1 = E_m \cdot \sin(pt - \alpha) / \sqrt{r_1^2 + L_1^2 p^2}$$

and
$$C_2 = pE_m k_2 \cdot \cos(pt - \beta) / \sqrt{1 + k_2^2 p^2 r_2^2},$$

where $\tan\alpha = L_1 p / r_1$, and $\tan\beta = pr_2 k_2$.

In many practical problems r_2 is extremely small, so that β is negligible.

If the terminals of a generator of electromotive force $E \equiv E_m \cdot \sin pt$, of self-inductance L, and of resistance r, be connected (Fig. 115) to the ends of a coil of resistance r_1 and self-inductance L_1, and if the coil ends are attached by leads of resistance r_2 to the coatings of a condenser of capacity k_2, we have

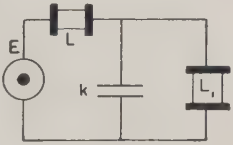

FIG. 115.

$$[(L + L_1)D_t + (r + r_1)]C_1 + (L \cdot D_t + r)C_2 = E_m \cdot \sin pt,$$

$$[L \cdot D_t^2 + r \cdot D]C_1$$
$$+ [L \cdot D_t^2 + (r + r_2) D_t + 1 / k_2]C_2 = pE_m \cdot \cos pt.$$

If $L = 0$, we have the case last considered.

If the terminals of a generator of resistance r and electromotive force $E_m \cdot \sin pt$ are connected (Fig. 116) by two conductors in parallel having resistances r_1, r_2; capacities k_1, k_2, and self-inductances L_1, L_2 respectively, but no mutual inductance,

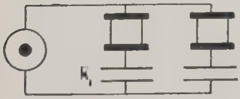

FIG. 116.

$$L_1 \cdot D_t^2 C_1 + (r + r_1)D_t C_1 + r \cdot D_t C_2 + C_1 / k_1 = pE_m \cdot \cos pt,$$

$$L_2 \cdot D_t^2 C_2 + r \cdot D_t C_1 + (r + r_2)D_t C_2 + C_2 / k_2 = pE_m \cdot \cos pt.$$

If we apply the operator $[L_2 \cdot D_t^2 + (r + r_2)D_t + 1 / k_2]$ to the first of these equations and the operator $[r \cdot D_t]$ to the second and subtract one result from the other, we shall have eliminated C_2 and may solve for C_1 in the usual manner.

In a case which sometimes occurs in practice, r is negligible and

$$C_1 = E_m \cdot \sin(pt - \alpha_1)/R_1, \quad C_2 = E_m \cdot \sin(pt - \alpha_2)/R_2,$$

where

$$R_1^2 = r_1^2 + (pL_1 - 1/k_1 p)^2, \; R_2^2 = r_2^2 + (pL_2 - 1/k_2 p)^2,$$

$$\tan \alpha_1 = (p^2 k_1 L_1 - 1)/pk_1 r_1, \quad \tan \alpha_2 = (p^2 k_2 L_2 - 1)/pk_2 r_2.$$

The reader will find the subject of this section fully discussed in Bedell and Crehore's *Alternating Currents*, Franklin and Williamson's *Elements of Alternating Currents*, Steinmetz's *Alternating Current Phenomena*, Heaviside's *Electrical Papers*, and in many other books.

87. Variable and Alternate Currents in Neighboring Circuits. If the coefficients of self-induction of two neighboring circuits s_1, s_2, — which contain constant generators the electromotive forces of which are E_1 and E_2 respectively, — are L_1, L_2, and their coefficient of mutual induction M, if the resistances of the circuits are r_1, r_2, and the currents which pass through them at the time t are C_1, C_2, then

$$L_1 \cdot D_t C_1 + M \cdot D_t C_2 + r_1 C_1 = E_1,$$

$$M \cdot D_t C_1 + L_2 \cdot D_t C_2 + r_2 C_2 = E_2.$$

It is to be noticed that, since the electrokinetic energy

$$\tfrac{1}{2} L_1 C_1^2 + M C_1 C_2 + \tfrac{1}{2} L_2 C_2^2,$$

or

$$\tfrac{1}{2} L_1 [(C_1 + M C_2 / L_1)^2 + C_2^2 (L_1 L_2 - M^2)/L_1^2],$$

must always be positive whatever the values or directions of the currents, if they exist at all, $L_1 L_2 - M^2$ can never be negative. If we substitute in the differential equations just found C_1' and C_2' for C_1 and C_2, where

$$C_1' \equiv C_1 - E_1/r_1, \quad C_2' \equiv C_2 - E_2/r_2,$$

we get

$$L_1 \cdot D_t C_1' + M \cdot D_t C_2' + r_1 C_1' = 0, M \cdot D_t C_1' + L_2 \cdot D_t C_2' + r_2 C_2' = 0,$$

or, symbolically written,

$$(L_1 \cdot D_t + r_1)\, C_1' + (M \cdot D_t)\, C_2' = 0, \quad (M \cdot D_t)\, C_1' + (L_2 \cdot D_t + r_2)\, C_2' = 0.$$

If we perform the operation $(L_2 \cdot D_t + r_2)$ on the first of these equations and the operation $(M \cdot D_t)$ on the second and subtract one of the resulting equations from the other, we shall eliminate C_2' and get the homogeneous linear equation

$$(L_1 L_2 - M^2)\, D_t^2 C_1' + (r_2 L_1 + r_1 L_2)\, D_t C_1' + r_1 r_2 C_1' = 0.$$

The general solution of this equation is of the form $A_1 e^{\lambda t} + B_1 e^{\mu t}$, where λ and μ are the two roots of the equation

$$(L_1 L_2 - M^2)\, x^2 + (r_2 L_1 + r_1 L_2)\, x + r_1 r_2 = 0,$$

that is,

$$\frac{-(r_2 L_1 + r_1 L_2) \pm \sqrt{(r_2 L_1 + r_1 L_2)^2 - 4\, r_1 r_2\, (L_1 L_2 - M^2)}}{2\,(L_1 L_2 - M^2)}.$$

If we eliminate C_1' from the original equations, we shall learn that $C_2' = A_2 e^{\lambda t} + B_2 e^{\mu t}$ where λ and μ have the values just given. Both λ and μ are negative, since $L_1 L_2 - M$ is positive, and both are real, since the expression under the radical sign may be written $(L_1 r_2 - L_2 r_1)^2 + 4 r_1 r_2 M^2$. The coefficients A_1, A_2, B_1, B_2 in the expression for C_1', C_2' are not all independent, for we find when we substitute these expressions in either of the original equations that the ratios A_2/A_1, B_2/B_1 must have the fixed values

$$-M\lambda/(L_2\lambda + r_2) \quad \text{or} \quad -(L_1\lambda + r_1)/M\lambda$$

and

$$-M\mu/(L_2\mu + r_2) \quad \text{or} \quad -(L_1\mu + r_1)/M\mu$$

respectively. If we denote these ratios by α and β, we have $C_1 = E_1/r_1 + A_1 e^{\lambda t} + B_1 e^{\mu t}$, $C_2 = E_2/r_2 + \alpha A_1 e^{\lambda t} + \beta B_1 e^{\mu t}$, where $\lambda, \mu, \alpha, \beta$ depend only upon the forms of the circuits and the materials of which they are made and A_1, B_1 are to be determined from the conditions of the particular problem under consideration.

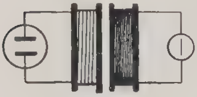

FIG. 117.

If $E_2 = 0$ (Fig. 117), and if s_1 be suddenly closed at the time $t = 0$, C_1 and C_2 are initially zero,

$$A_1 + B_1 = -E_1/r_1, \quad A_1\alpha + B_1\beta = 0,$$

$$A_1 = \beta E_1/r_1(\alpha - \beta), \quad B_1 = \alpha E_1/r_1(\beta - \alpha),$$

and
$$C_1 = E_1/r_1\{1 - \tfrac{1}{2}e^{\lambda t}[(r_2L_1 - r_1L_2)/R + 1] + \tfrac{1}{2}e^{\mu t}[(r_2L_1 - r_1L_2)/R - 1]\},$$

$$C_2 = E_1M(e^{\mu t} - e^{\lambda t})/R,$$

where R stands for the radical in the expressions for λ and μ. The time integral of C_2 from 0 to ∞ is evidently

$$E_1M(\mu - \lambda)/\mu\lambda R \text{ or } -E_1M/r_1r_2,$$

and is the same whichever circuit contains the given electromotive force and is used as the primary. Fig. 118 shows the currents in s_1 and s_2 under these circumstances, when

$$E_1 = 2, L_1 = \tfrac{1}{2}, L_2 = \tfrac{1}{2},$$
$$M = 1/\sqrt{8}, r_1 = 1, r_2 = 1.$$

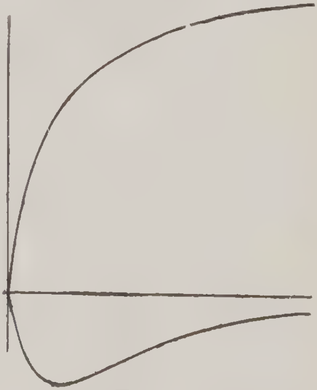

FIG. 118.

If $E_2 = 0$, and if s_1, which has been closed for some time, has its resistance suddenly changed, when $t = 0$, from r_0 to r_1, we have initially $C_1 = E_1/r_0$ and $C_2 = 0$, so that

$$A_1 + B_1 = E_1(r_1 - r_0)/r_0r_1,$$
$$A_1\alpha + B_1\beta = 0,$$

$$A_1 = \beta E_1(r_0 - r_1)/r_0r_1(\alpha - \beta), \quad B_1 = \alpha E_1(r_1 - r_0)/r_0r_1(\alpha - \beta),$$

and
$$C_2 = E_1(r_1 - r_0)(e^{\mu t} - e^{\lambda t})\alpha\cdot\beta/r_0r_1(\alpha - \beta).$$

The integral of C_2 with respect to the time, from 0 to ∞, is $E_1 M(r_1 - r_0)/r_0 r_1 r_2$, and the limit of this, as r_1 grows larger without limit, is $E_1 M/r_0 r_2$. C_2 attains its maximum value at the time $(\log \lambda/\mu)/(\mu - \lambda)$: this fraction approaches zero when r_1 increases without limit. If r_1 is infinite, we have $C_1 = 0$, $C_2 = Ae^{-r_2 t}/L_2$: the time integral of C_2, between 0 and ∞, is AL_2/r_2, from which we infer that $A = E_1 M/r_0 L_2$.

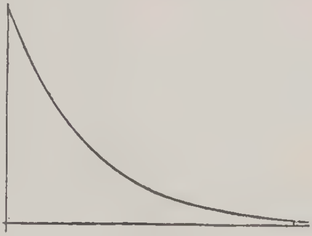

FIG. 119.

If, then, r_1 is infinite, that is, if the circuit is suddenly broken, the current in the secondary jumps instantly from zero to the value $E_1 M/r_0 L_2$ and then decreases after the manner shown in Fig. 119, which is drawn to scale on the assumption that in practical units $E_1 = 2$, $L_2 = \frac{1}{2}$, $M = 1/\sqrt{8}$, $r_0 = 1$. The whole area between the time axis and the curve which represents the current in s_2 is the same in Fig. 119 as in Fig. 118, though the shapes of the curves are very different.

If $E_2 = 0$, and if $E_1 = E_m \cdot \cos pt$, we have

$$L_1 \cdot D_t C_1 + M \cdot D_t C_2 + r_1 C_1 = E_m \cdot \cos pt$$

and
$$M \cdot D_t C_1 + L_2 \cdot D_t C_2 + r_2 C_2 = 0.$$

If $C_1 \equiv x$, $C_2 \equiv y$ represent any special solution of these equations, the complete solution may be found by adding x and y to the values of C_1 and C_2, found earlier in this section, which completely solve the equations formed by equating to zero the first members. These last quantities, however, are exponential in form with indices intrinsically negative; after a few seconds they are negligible, and we need use only the final forms of x and y. The real parts (u_1, u_2) of any solution, of the form $C_1 = u_1 + v_1 i$, $C_2 = u_2 + v_2 i$, of the equations

$$L_1 \cdot D_t C_1 + M \cdot D_t C_2 + r_1 C_1 = E_m e^{pti}, M \cdot D_t C_1 + L_2 \cdot D_t C_2 + r_2 C_2 = 0,$$

form a solution of the original equations. Applying the operator $(L_2 \cdot D_t + r_2)$ to the first of these new equations and the operator $(M \cdot D_t)$ to the second and subtracting one result from the other, we get

$$(L_1L_2 - M^2)\,D_t^2C_1 + (r_1L_2 + r_2L_1)\,D_tC_1 + r_1r_2C_1$$
$$= E_m(L_2pi + r_2)\,e^{pti},$$

an equation which evidently has a solution of the form $B \cdot e^{pti}$, and if we substitute this expression in the equation, we learn that

$$B = E_m(r_2 + L_2pi)\,/\,[r_1r_2 + p^2M^2 - p^2L_1L_2 + (r_2L_1 + r_1L_2)\,pi].$$

The real part (x) of Be^{pti} is, therefore,

$$\frac{E_m \cdot \sqrt{L_2^{\,2}p^2 + r_2^{\,2}} \cdot \cos(pt + \delta - \theta)}{\sqrt{[r_1r_2 - (L_1L_2 - M^2)\,p^2]^2 + (r_2L_1 + r_1L_2)^2p^2}},$$

where $\tan\delta = L_2p\,/\,r_2$ and

$\tan\theta = (r_2L_1 + r_1L_2)p\,/\,[r_1r_2 - (L_1L_2 - M_2)p_2];$

or $x \equiv A \cdot \cos(pt - \alpha)$, where, if

$L \equiv L_1 - M^2L_2p^2\,/\,(L_2^{\,2}p^2 + r_2^{\,2})$

and $r \equiv r_1 + M^2p^2r_2\,/\,(L_2^{\,2}p^2 + r_2^{\,2}),$

$A = E_m\,/\,\sqrt{L^2p^2 + r^2}$, and $\tan\alpha = Lp\,/\,r.$

The primary, therefore, behaves like a single circuit (at a distance from all others) of resistance r, greater than r_1, and of self-inductance L, less than L_1. The presence of the secondary circuit makes the lag in the primary less than it would otherwise be.

The corresponding value (y) of C_2 can be found by substituting x in one of the original equations: it is

$$-MpA \cdot \cos(pt - \beta)\,/\,\sqrt{L_2^{\,2}p^2 + r_2^{\,2}},$$

or

$$MpA \cdot \cos(pt + \pi - \beta)\,/\,\sqrt{L_2^{\,2}p^2 + r_2^{\,2}},$$

where tan $(\beta - \alpha) = r_2/L_2p$. The lag in phase of the secondary circuit behind the primary is $\pi + \alpha - \beta$, or $\frac{1}{2}\pi + \tan^{-1}(L_2p/r_2)$. The lag of the secondary current behind the electromotive force is $\tan^{-1}[p^2(L_1L_2 - M^2) - r_1r_2]/[p(r_2L_1 + r_1L_2)]$. The average rate for any number of whole periods at which the generator furnishes energy to the primary is the average value of $E_mA \cdot \cos pt \cdot \cos(pt - \alpha)$, which is $\frac{1}{2}E_mA \cdot \cos\alpha$ or $E_m{}^2r/2(L^2p^2 + r^2)$; this is greater when the secondary is closed than when it is open. The average rate for any whole number of periods at which energy is used in heating the secondary is the average value of $C_2{}^2 r_2$ or $r_2M^2p^2A^2/2(L_2{}^2p^2 + r_2{}^2)$; the ratio of this to the power used in the primary is called the *efficiency* of the transformation and is equal to $r_2M^2p^2/r(L_2{}^2p^2 + r_2{}^2)$. The electromotive force induced in the secondary is

$$-L_2 \cdot D_tC_2 - M \cdot D_tC_1.$$

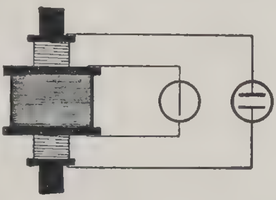

FIG. 120.

The problem here considered is in principle that of the alternate current transformer (Figs. 120 and 121), and it is frequently the case in practice that the ratio of r_2 to L_2p is very small. Under these circumstances L, r, the amplitude of C_2, and $\beta - \alpha$ are nearly equal to

$$L_1 - M^2/L_2,\ r_1 + r_2M^2/L_2{}^2,\ MA/L_2, \text{ and } 0$$

respectively. Both circuits are usually wound on a soft iron core (often a ring) of great permeability, and $L_1L_2 - M^2$ is very small compared with either L_1 or L_2; in this case the lag of the primary is negligible, while, for high frequencies, that of the secondary is nearly two right angles. If $L_1L_2 - M^2$ is practically nothing, the transformer is said to have no magnetic leakage. The ratio of L_1 to L_2

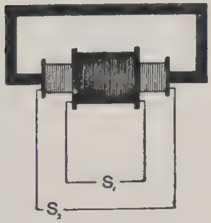

FIG. 121.

is usually nearly equal to that of the square of the number of turns (n_1^2/n_2^2) of the circuits on the core, and under these circumstances B is approximately equal to n_1A/n_2. For exhaustive treatments of the problem of this section, which is of much practical importance, the reader is referred to such books as Fleming's *The Alternate Current Transformer;* J. J. Thomson's *Elements of Electricity and Magnetism;* Nipher's *Treatise on Electricity and Magnetism;* and Steinmetz's *Alternating Current Phenomena.*

88. The General Equations of the Electromagnetic Field. When a fixed, metallic, linear circuit s of specific conductivity $\lambda \equiv 1/\sigma$, at a uniform temperature throughout, carries an induced current, positive electricity is urged around s in the direction of the current by "something of the nature of an electrostatic field," though we do not need to assume that this is always due to electrostatic charges. If we denote the components of the field, at every point within or without the conductors which form the circuit by X, Y, Z, the line integral of $[X \cdot \cos(x, s) + Y \cdot \cos(y, s) + Z \cdot \cos(z, s)]$, taken around s in the direction of the current, is the internal electromotive force and is equal to the negative of the time rate of change of the positive flux of magnetic induction through the circuit. If the circuit be covered by a cap S, if n denotes the direction of the normal to S drawn towards the positive side, and if B_x, B_y, B_z are the components of the magnetic induction B, then, on the assumption that Stokes's Theorem may be applied to the vector (X, Y, Z), we shall have

$$\iint [(D_yZ - D_zY)\cos(x, n) + (D_zX - D_xZ)\cos(y, n) \\ + (D_xY - D_yX)\cos(z, n)]\,dS$$

$$= -\iint [D_tB_x \cdot \cos(x, n) + D_tB_y \cdot \cos(y, n) \\ + D_tB_z \cdot \cos(z, n)]\,dS,$$

so that the expression

$$[(D_yZ - D_zY + D_tB_x)\cos(x,n) + (D_zX - D_xZ + D_tB_y)\cos(y,n) \\ + (D_xY - D_yX + D_tB_z)\cos(z, n)]$$

integrated over any cap bounded by s, whatever the forms of the latter, yields zero. We are led to assume, therefore, that at every point within or without any such circuit

$$\begin{aligned} -D_tB_x &= D_yZ - D_zY, \\ -D_tB_y &= D_zX - D_xZ, \\ -D_tB_z &= D_xY - D_yX, \end{aligned} \qquad [209]$$

and to say that the negative of the vector the components of which are the time derivatives of the component of the induction is equal to the curl of the electric field.

If ξ, η, ζ are the components of the curl of the magnetic induction B, and if the components F_x, F_y, F_z of the vector F are defined by the equations $4\pi F_x = \text{Pot}\,\xi$, $4\pi F_y = \text{Pot}\,\eta$, $4\pi F_z = \text{Pot}\,\zeta$, F is a vector potential function of B. By its aid we can transform the integral

$$-\int\!\!\int [D_tB_x \cdot \cos(x, n) + D_tB_y \cdot \cos(y, n) + D_tB_z \cdot \cos(z, n)]\,dS,$$

in which the integrand is the component normal to S of the curl of D_tF, into a line integral taken about s of the tangential component of D_tF. We have, therefore,

$$\int [X \cdot \cos(x, s) + Y \cdot \cos(y, s) + Z \cdot \cos(z, s)]\,ds$$

$$= -\int [D_tF_x \cdot \cos(x, s) + D_tF_y \cdot \cos(y, s) + D_tF_z \cdot \cos(z, s)]\,ds,$$

and the integrands can differ only by the tangential component of some lamellar vector (G_x, G_y, G_z), which adds nothing to the integral taken completely around s. Since this is true whatever the shape of s, we assume that at every point

$$X = -D_tF_x + G_x, \quad Y = -D_tF_y + G_y, \quad Z = -D_tF_z + G_z.$$

When the magnetic field is constant and the components of D_tF vanish, X, Y, Z are equal to $-D_xV$, $-D_yV$, $-D_zV$, and the phenomena will be accounted for if we follow Maxwell and write

$$X = -D_tF_x - D_xV, \quad Y = -D_tF_y - D_yV,$$
$$Z = -D_tF_z - D_zV. \qquad [210]$$

The reader should compare these equations with [208].

Within the conductors which form s, the components (u, v, w) of the conduction current (q) satisfy Maxwell's current equations

$$4\pi u = D_yN - D_zM, \ 4\pi v = D_zL - D_xN,$$
$$4\pi w = D_xM - D_yL, \qquad [211]$$

where L, M, N are the components of the magnetic field, and $u = \lambda X$, $X = \sigma u$, $Y = \sigma v$, $Z = \sigma w$.

According to Poisson's hypothesis, a dielectric consists of perfectly conducting molecules separated from each other by perfectly insulating spaces, the specific inductive capacity (K) depending merely upon the ratio of the volumes of the spaces occupied by the molecules and the intervening spaces. From this point of view, there is a transfer of electricity through every molecule when the dielectric is being polarized, one portion of the surface of the molecule becoming positively electrified by induction and another portion negatively electrified. Every change in the polarization is accompanied by the passage of electricity through the mass of the molecule, and we are to assume that during the change every molecule acts electromagnetically like a current element. Whatever our theory, the appearance of the induced charges which account mathematically for the phenomena observed when a dielectric becomes polarized, involves the displacement of electricity, and corresponding electromagnetic effects. In his famous paper on "A Dynamical Theory of the Electromagnetic Field," published in the *Philosophical Transactions of the Royal Society* in 1864, Maxwell assumed that whenever the polarization of a soft dielectric in which the electric induction has the components Φ_x, Φ_y, Φ_z is being changed, electromagnetic phenomena are to be looked for equivalent to those which would accompany the presence of currents, called *displacement currents*, in the dielectric defined at each point by the vector

$$(D_t\Phi_x/4\pi,\ \ D_t\Phi_y/4\pi,\ \ D_t\Phi_z/4\pi)$$

or $$(K\cdot D_tX/4\pi,\ \ K\cdot D_tY/4\pi,\ \ K\cdot D_tZ/4\pi).$$

According to this assumption,

$$u' = D_t\Phi_x/4\pi + \lambda X,\quad v' = D_t\Phi_y/4\pi + \lambda Y,$$
$$w' = D_t\Phi_z/4\pi + \lambda Z,$$

where u', v', w' are the components of the total current, and we may write the current equations in the generalized form

$$\begin{aligned} 4\pi u' &\equiv D_t\Phi_x + 4\pi u = D_yN - D_zM,\\ 4\pi v' &\equiv D_t\Phi_y + 4\pi v = D_zL - D_xN,\\ 4\pi w' &\equiv D_t\Phi_z + 4\pi w = D_xM - D_yL, \end{aligned} \qquad [212]$$

in which u, v, w represent the components of the conduction current alone. In conductors the displacement currents are negligible, in a perfectly insulating dielectric the conduction currents vanish; both are supposed to coexist in dielectrics which are slightly conducting. Within a conductor, since the curl of the magnetic force is solenoidal, $D_xu + D_yv + D_zw = 0$.

If at least that portion of the magnetic induction near the current which changes with the time, is induced in soft media, and if μ is the magnetic inductivity at the point (x, y, z), we have $D_tB_x = \mu\cdot D_tL$, $D_tB_y = \mu\cdot D_tM$, $D_tB_z = \mu\cdot D_tN$, and [209] becomes

$$-\mu\cdot D_tL = D_yZ - D_zY,\ \ -\mu\cdot D_tM = D_zX - D_xZ,$$
$$-\mu\cdot D_tN = D_xY - D_yX, \qquad [213]$$

or, if the media are homogeneous,

$$-\mu\lambda\cdot D_tL = D_yw - D_zv,\ \ -\mu\lambda\cdot D_tM = D_zu - D_xw,$$
$$-\mu\lambda\cdot D_tN = D_xv - D_yu. \qquad [214]$$

If we differentiate the equations of [212] with respect to t and substitute the values of D_tL, D_tM, D_tN from [214] in the results, we shall get for homogeneous media three equations of the form

$$\mu\lambda(K \cdot D_t^2X + 4\pi \cdot D_tu) = \nabla^2u - D_x(D_xu + D_yv + D_zw) = \nabla^2u,$$

that is,

$$\mu\lambda(K \cdot D_t^2X + 4\pi \cdot D_tu) = \nabla^2u, \quad \mu\lambda(K \cdot D_t^2Y + 4\pi \cdot D_tv) = \nabla^2v,$$
$$\mu\lambda(K \cdot D_t^2Z + 4\pi \cdot D_tw) = \nabla^2w. \qquad [215]$$

Where there is no conduction current these become

$$\mu K \cdot D_t^2X = \nabla^2X, \; \mu K \cdot D_t^2Y = \nabla^2Y, \; \mu K \cdot D_t^2Z = \nabla^2Z. \qquad [216]$$

If we substitute in the equations [214] the values of u, v, and w from [211], we shall obtain for homogeneous media the equations

$$4\pi\mu\lambda \cdot D_tL = \nabla^2L, \qquad 4\pi\mu\lambda \cdot D_tM = \nabla^2M,$$
$$4\pi\mu\lambda \cdot D_tN = \nabla^2N. \qquad [217]$$

The energy of the field is $W + T$ where

$$W \equiv \frac{1}{8\pi}\iiint_\infty K(X^2 + Y^2 + Z^2)\,d\tau,$$

$$T \equiv \frac{1}{8\pi}\iiint_\infty \mu(L^2 + M^2 + N^2)\,d\tau.$$

MISCELLANEOUS PROBLEMS.

1. The astronomical unit of mass in any length-mass-time system is the mass which, concentrated at a fixed point, would cause by its attraction unit acceleration in any particle at the unit distance. The astronomical unit of mass concentrated at a point at a unit distance from a particle of mass equal to the absolute unit would attract it with a force of one unit. Show that the astronomical unit of mass in the c.g.s. system is 15,430,000 grammes, while in the f.p.s. system it is 963,000,000 pounds. Show also that the mass which, concentrated at a point distant 1 centimetre from a particle of equal mass, would attract it with a force of 1 dyne, is only 3928 grammes. Prove that the earth's mass (Problem 9) in astronomical c.g.s. units is 3.98×10^{20}. Show that a mass of 1 kilogramme must be raised about 3 metres at the earth's surface in order to reduce its weight by 1 dyne.

2. Prove that two equal marbles, each of 4 grammes mass, must be placed with centres a little over 1 centimetre apart, if the attraction between them is to be 1 microdyne, and find the attraction [$5535\,k\pi$] of an iron cylinder of revolution, of 10 centimetres radius, 1 metre long, upon a marble of 100 grammes mass, with centre in the axis of the cylinder and distant 10 centimetres from the nearer base. If the specific gravity of iron is 7.5, the radius of each of two equal iron balls which, placed in contact, attract each other with a force of one gramme's weight is 88.5 centimetres. If the mass of each of two equal homogeneous spheres with centres 1 mile apart were 415,000 gross tons, the attraction between them would

be about 1 pound's weight. The force of attraction between two equal particles 1 foot apart and each of mass n times as great as that of a cubic foot of water, would be equal to the weight of about $n^2/(7.94 \times 10^6)$ pounds.

3. Assuming that a force equivalent to the weight of a mass of 1 gramme is equal to $4^2\pi^2(98.95)^4$ centimetre-gramme attraction units, find the radii of two equal homogeneous spheres which, made of matter of density 6, would attract each other with a force of 1 gramme's weight if they were placed in contact with each other. [98.95.]

4. Assuming that 1 dyne is equal to 15,430,000 absolute c.g.s. attraction units and that 1 poundal is equal to 13,825 dynes, show that if two equal homogeneous spheres of density ρ, when placed in contact, attract each other with a force of f dynes, the radius of each is about $(43.3)\sqrt[4]{\dfrac{f}{\rho^2}}$ cm., and that two equal homogeneous spheres of the density of water when in contact will attract each other with a force of 1 dyne, 1 gramme's weight, 1 poundal, or 1 pound's weight, according as the radius of each in centimetres is 43.3, 242.2, 469.4, or 1118.5.

5. Show that, having found the value of the attraction unit of force in any length-mass-time system in terms of the absolute unit of force in this system, you may find the value of the attraction unit of force in any other system the ratios of the fundamental units of which to those of the old system are λ, μ, and τ, by multiplying the found value by $\dfrac{\mu\tau^2}{\lambda^3}$.

6. Show that if two homogeneous spheres of mass m_1 and m_2, starting from rest with centres at a distance a apart, move toward each other under their mutual attraction, and if at any time t, x represents the distance between the centres,

$$D_t^2 x = \frac{-k\,(m_1 + m_2)}{x^2}, \quad D_t x = -\sqrt{\frac{2\,k\,(m_1 + m_2)\,(a - x)}{ax}},$$

$$t=\sqrt{\frac{a}{2\,k\,(m_1+m_2)}}\left\{\sqrt{x\,(a-x)}+a\cos^{-1}\sqrt{\frac{x}{a}}\right\}$$

$$=\sqrt{\frac{a}{2\,k\,(m_1+m_2)}}\left\{\sqrt{x\,(a-x)}+a\tan^{-1}\sqrt{\frac{a-x}{x}}\right\}.$$

Hence prove that if the spheres are each one foot in diameter and of density equal to the earth's mean density, and if their surfaces are $\frac{1}{4}$ of an inch apart at the start, they will come together in about five minutes and a half. In this connection we may note that if M is the mass of the earth, R its radius, ρ its mean density, and k the gravitation constant for the particular units used,

$$g=\frac{kM}{R^2}\text{ and }\rho=\frac{3\,g}{4\,\pi Rk}.$$

If the first sphere is fixed while the second, of mass m_2, is free to move,

$$D_t^2x=\frac{-\,km_1}{x^2},\quad D_tx=-\sqrt{\frac{2\,km_1\,(a-x)}{ax}},$$

$$t=\sqrt{\frac{a}{2\,km_1}}\left\{\sqrt{x\,(a-x)}+a\cos^{-1}\sqrt{\frac{x}{a}}\right\}.$$

If in this case the radius of the fixed sphere is r, and if m_2 is comparatively small and a infinite, the velocity with which the second sphere reaches the surface of the first is sometimes called the *final velocity* for bodies falling to the fixed sphere. Its value is $\sqrt{\frac{2\,km_1}{r}}$, or $\sqrt{2f\cdot r}$, where f is the force of gravitation at the surface of the fixed sphere.

Show that if the diameter of the sun is 109.4 times that of the earth and its mass 331,100 times the earth's mass, the final velocity for bodies falling into the sun is 55 times the final velocity for bodies falling into the earth. The radius of the earth being 6.37×10^8 centimetres, show that the final velocity for bodies falling to the earth under the attraction of the earth only is nearly 11,180 metres (or about 7 miles) per second.

7. Show that if a meteor falls upon a planet with velocity equal to that which it would acquire if it fell from rest at an infinite distance from the planet under the planet's attraction, its kinetic energy will be proportional to the product of the radius of the planet and the force of gravity on its surface.

8. Given that a falling body reaches the earth's surface with a velocity v_0, compute the height through which it has fallen from rest, first, on the assumption that the force which urged it was constant, and, secondly, on the assumption that the force varied inversely as the square of the distance of the body from the earth's centre, and prove that the difference between the reciprocals of the answers you obtain is equal to the reciprocal of the earth's radius.

9. Given the radius of the earth in centimetres (6.37×10^8), the mass of the earth in grammes (6.14×10^{27}), the radius of the sun (6.97×10^{10}), the mass of the sun (2.03×10^{33}), and the mean distance between the centres of the earth and sun (1.49×10^{13}), find the time when the sun and earth would come together, if both were arrested in their paths. Prove that the acceleration due to gravity is at the sun's surface about $27.6\,g$.

10. A body of mass m falls from rest near the surface of the earth and is retarded by the resistance of the air, which is λv^2 dynes when the velocity is v. Show that if s represents the space passed over up to the time t, and if $\mu \equiv \lambda / m$ and $c^2 \equiv g / \mu$, $2\,\mu ct = \log[(c + v) / (c - v)]$, $2\,\mu s = \log[c^2 / (c^2 - v^2)]$, $v^2 = c^2(1 - e^{-2\mu s})$, and $\mu s = \log\cosh(\mu ct)$.

Show that if the body were thrown upward with initial velocity v_0, we should have $\tan(\mu ct) = c(v_0 - v) / (c^2 + v_0 v)$.

If in the case of the falling body v is the actual velocity and v' the velocity which would be required by falling through the same distance *in vacuo*,

$$v^2 / v'^2 = 1 - \tfrac{1}{2}\, v'^2 / c^2 + \tfrac{1}{6}\, v'^4 / c^4 - \tfrac{1}{24} v'^6 / c^6 + \cdots.$$

11. Show that the periodic time of a planet moving about a fixed sun of mass m in a circular orbit of radius r is $2\pi r^{\frac{3}{2}}/\sqrt{km}$, where $1/k$ is the ratio of the absolute unit of force in the given length-mass-time system to the corresponding attraction unit; and, assuming that the diminution of gravity at the equator due to the earth's rotation is about $\frac{1}{289}$th of the whole, and that the mean distance of the moon from the earth's centre is about 60 times the earth's radius, compute the length of the month.

12. When a particle moves in any plane curve, the tangential and interior normal acceleration components are $D_t v$ and v^2/ρ, while the acceleration components, taken along and perpendicular to the radius vector which joins any fixed point in the plane used as the origin of a system of polar coördinates, to the particle, are $D_t^2 r - r(D_t\theta)^2$ and $D_t(r^2 \cdot D_t\theta)/r$ respectively. If the resultant acceleration is always directed towards the origin, $D_t(r^2 D_t\theta) = 0$ and $r^2 \cdot D_t\theta = h$, so that the areas of the sectors swept over in any two time intervals by the radius vector are to each other as the lengths of the intervals: if p represents the perpendicular let fall from the origin upon the tangent to the path, $vp = r^2 D_t\theta = h$.

The acceleration towards the origin is

$$R = r(D_t\theta)^2 - D_t^2 r,$$

and, if u represents the reciprocal of r, this may be written

$$h^2u^2(u + D_\theta^2 u).$$

Since
$$v^2 = h^2\left[u^2 + (D_\theta u)^2\right],$$

$$\tfrac{1}{2} D_t(v)^2 = h^2 D_t u\,(u + D_\theta^2 u) = -R \cdot D_t r.$$

In the case of a planet describing a plane orbit about a fixed primary centred at the origin

$$R = \mu^2 u^2 = h^2u^2(u + D_\theta^2 u);\ \text{or } D_\theta^2 z + z = 0,$$

where
$$z = u - \frac{\mu^2}{h^2},\ \text{so that } u = \frac{\mu^2}{h^2} + C\sin(\theta - \lambda).$$

This is the equation of a conic section referred to a focus as origin: if e is the eccentricity and m the distance of the focus from the directrix, $C = -1/m$ and $h^2/\mu^2 = em$. The angle ψ between the radius vector, drawn from the origin to any point on the orbit and the tangent at the point, is given by the equation, $\text{ctn}\,\psi = -r \cdot C \cdot \cos(\theta - \lambda)$. Assuming that, when θ is zero, $\psi = \alpha$, $r = a$, and $v = v_0$, show that $h = v_0 a \cdot \sin\alpha$, and $1 - e^2 = (2\mu^2 - v_0^2 a)\,h^2/a\mu^4$. Discuss separately the three cases where v_0^2 is respectively less than, equal to, and greater than $2\mu^2/a$, and find the lengths of the semiaxes of the orbit. Show that, if $\alpha = 90°$ and if $v_0^2 a = \mu^2$, the orbit will be circular; show also that, if T is the periodic time of the planet and a the semiaxis major of its orbit, $\mu^2 T^2 = 4\pi^2 a^3$.

13. Assuming that the equation

$$t = \sqrt{\frac{a}{g}} \int_0^\phi \frac{d\phi}{\sqrt{1 - \sin^2 \frac{1}{2}\alpha \sin^2 \phi}} \equiv \sqrt{\frac{a}{g}} \cdot F\,(\sin \tfrac{1}{2}\alpha, \phi),$$

where $\sin\phi \cdot \sin\frac{1}{2}\alpha = \sin\frac{1}{2}\theta$, and α is the angular amplitude on one side of the vertical, gives the time occupied by a simple pendulum of length a in going from the vertical position to a position in which the thread makes the angle θ with the vertical; and that the complete time of swing is

$$2\pi\sqrt{\frac{a}{g}}\left[1 + \tfrac{1}{4}\sin^2\tfrac{1}{2}\alpha + \tfrac{9}{64}\sin^4\tfrac{1}{2}\alpha + \cdots\right];$$

assuming also that a rigid body swinging about a horizontal axis under gravity moves like a simple pendulum of length k^2/h where h is the distance of the centre of gravity from the axis and k is the radius of gyration; show how a pendulum may be used to measure the force of gravity at a point.

If the earth were a homogeneous sphere, would a clock which at a given temperature keeps correct time on the earth's surface lose or gain at the same temperature at the bottom of a deep mine? Assuming that if g_λ and g_0 are the accelerations due to gravity at sea level, in latitude λ and at the equator respectively $g_\lambda = g_0\,(1 + .005226\,\sin^2\lambda)$ and $g_0 = 978.1$; show that the lengths of the seconds pendulum at the north pole, in latitude 45°, and at the equator, are about 99.6 centimetres, 99.3 centimetres, and 99.1 centimetres.

A pendulum which beats seconds on the earth's surface gains n seconds per day in a mine h metres deep. Show that if ρ_0 is the mean density of the earth and ρ the density of the surface stratum,

$$\frac{n}{86400} = \frac{\pi h}{4 \cdot 10^7}\left(2 - \frac{3\,\rho}{\rho_0}\right) \text{ approximately.}$$

14. Assuming that the earth is a homogeneous sphere, of radius 6.37×10^8 centimetres and of mass 6.14×10^{27} grammes, rotating uniformly about its axis in 86164 seconds, so that the velocity of a point on the equator is about 463 metres per second, show that the angular velocity of the earth is 0.00007292 or about $(13713)^{-1}$ radians per second, and that the downward acceleration at the equator is by 3.39 centimetres per second per second, or about $\frac{G}{289}$, less than the acceleration, G, at the poles. Show also (Fig. 122) that the acceleration of gravity towards the earth's centre at the latitude λ is $G\left(1 - \frac{\cos^2\lambda}{289}\right)$, the deviation of the plumb line $\tan^{-1}\left(\frac{\sin\lambda\cos\lambda}{289 - \cos^2\lambda}\right)$, and the horizontal component of apparent gravitation $\frac{G}{289}\sin\lambda\cos\lambda$.

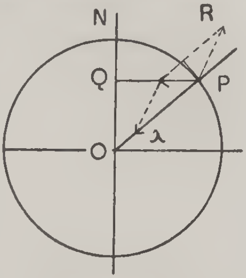

Fig. 122.

15. If in the case of any homogeneous spherical body rotating uniformly about its axis, the polar gravity acceleration and the equatorial gravity acceleration be g_p and g_e

respectively, the acceleration of gravity towards the earth's centre in latitude λ is $(g_p \sin^2\lambda + g_e \cos^2\lambda)$ and the deviation of the plummet from the geometrical vertical is

$$\tan^{-1}\left\{\frac{(g_p - g_e)\sin\lambda\cos\lambda}{g_p\sin^2\lambda + g_e\cos^2\lambda}\right\}.$$

16. A bicycle and its rider weigh together 75 kilogrammes. Show that if the machine were driven first eastward and then westward in this latitude at a velocity of 10 metres per second, the difference between the pressures on the ground in the two cases would be about 16.5 grammes.

17. The centre of a planet of radius a moves around a sun of mass M in a circular orbit of radius r. Compute the pressures exerted on the surface of the planet by two equal particles, each of mass m, situated respectively on the points of the planet nearest and farthest from the sun. Show that the difference between these pressures is small compared with the difference between the attractions of the sun upon these particles.

What is the difference between the apparent weights of a body of mass m on the earth's equator about September 21, at noon and at midnight?

18. Two rods AB and CD, both of line density ρ, are placed parallel to each other. Show that the force on either in the direction of its length is

$$\rho^2\left\{\log\frac{AC + AD + CD}{AC + AD - CD} - \log\frac{BC + BD + CD}{BC + BD - CD}\right\}.$$

The component of the mutual attraction perpendicular to the rods is $2\rho^2(BC - BD - AC + AD)/r$, where r is the perpendicular distance between them.

19. The sides of a triangle are formed of three thin uniform rods of equal density. Prove that a particle attracted by the sides is in equilibrium if placed at the centre of the inscribed circle. [M. T.]

20. Every particle of three similar, uniform rods of infinite length lying in the same plane, attracts with a force varying inversely as the square of the distance: prove that a particle subject to the attraction of the rods will be in equilibrium, if it be placed at the centre of gravity of the triangle enclosed by the rods. [M. T.]

21. The attraction of the straight rod AB at a point P is the resultant of two forces, each equal to f, acting at P towards the extremities of the rod,

where $$f \equiv 2m \cdot AB / [(AP + BP)^2 - AB^2].$$

Find the value of f when P lies on an ellipse the foci of which are the extremities of the rod. [Routh.]

22. If the direction at the point O of the attraction of every portion of a uniform plane curvilinear wire bisects the angle subtended at O by that portion, the wire is either straight or has the form of a circumference with centre at O. [Routh.]

23. If the law of attraction be the inverse square, two curvilinear rods in one plane exert equal attractions at the origin if the densities at points on the two rods on any radius vector drawn through the origin are proportional to the perpendiculars from the origin on the tangents. [Routh.]

24. Prove directly from the formula for the attraction of a slender straight wire, that the attraction at a point P, due to an infinite homogeneous cylinder of any form, is twice that of so much of the cylinder as is cut off by a double cone formed by the revolution about a line through P, parallel to the generating lines of the cylinder, of a line which cuts this line at P at an angle of 60°.

25. A uniform wire AB in the form of a circular arc has its centre at O. Prove that the component of the attraction, at any point P, in a direction perpendicular to the plane containing P and the normal at O to the plane of the arc, is $a\mu(r_1^{-1} - r_2^{-1})/h$, where $r_1 = AP$, $r = BP$, h is the projection of OP on the plane of the arc, and μ the line density of the wire.

26. Prove that the attraction in the direction PO at a point P on the circumference of a circle the centre of which is O, due to an infinitely long, straight filament of given density passing through a point Q in the circumference and perpendicular to its plane, is the same wherever the point Q is. If the filaments of a homogeneous columnar distribution of given mass per unit length are so arranged that the cross-section is a circle passing through a point P, the attraction of the distribution on P will be a maximum. [Tarleton.]

27. A water tower in the shape of a cylinder of revolution is 100 feet high and 10 feet in diameter. The mass of the tower and contents is 8400 pounds per foot of height. Without the help of pencil or paper, guess, to within one per cent of the truth, the value in f.p.s. attraction units of the horizontal component of the attraction due to the tower at a point at its foot just outside it.

28. Prove that at a point on the edge of an infinite homogeneous cylinder of semicircular cross-section, the components of the attraction across the plane face perpendicular to the axis, and normal to the face, are $\pi ak\mu$ and $2\,ak\mu$ respectively, and show that gravity is diminished by the fraction $\frac{3\,a}{\pi r}\cdot\frac{\rho-\rho'}{\rho_0}$ at the middle of the surface of a long straight canal of semicircular section, a being the radius of the semicircle, r the radius of the spherical earth, ρ' the density of water, ρ that of the surface stratum of the earth, and ρ_0 the earth's mean density. The corresponding quantity in the case of a canal of rectangular cross-section of depth a and breadth $2\,a$ is

$$\frac{\pi+2\cdot\log 2}{\pi}\cdot\frac{3\,a}{4\,r}\cdot\frac{\rho-\rho'}{\rho_0}.$$

29. An infinitely long homogeneous prism has a rectangular cross-section of length a and breadth b. Assuming that

$$\int\log(a^2+x^2)\,dx = x\cdot\log(a^2+x^2)-2\,x+2\,a\tan^{-1}(x/a),$$

show that at any point on one of the edges the components of the attraction along the sides a and b of the cross-section through the point are

$$k\rho\{2a\tan^{-1}(b/a)+b\cdot\log[(a^2+b^2)/b^2]\}$$

and $$k\rho\{2b\tan^{-1}(a/b)+a\cdot\log[(a^2+b^2)/a^2]\}.$$

If the ratio of b to a is large, the first of these quantities is nearly equal to $\pi a\rho k$. Show that the apparent latitude of a point on one edge of a long, deep, narrow crevasse of breadth a, running east and west, is altered by the angle $3\rho a/4\rho_0 r$, nearly, by the presence of the crevasse. [Thomson and Tait.]

30. Assuming that the attraction of a homogeneous cylinder of revolution, of density ρ, radius a, and height h, upon a unit particle at the centre of one of its ends, is

$$2\pi k\rho a\left[1-\frac{a}{2h}+\frac{1\cdot 1}{2\cdot 4}\cdot\frac{a^3}{h^3}-\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\cdot\frac{a^5}{h^5}+\cdots\right]$$

or $$2\pi k\rho h\left[1-\frac{h}{2a}+\frac{1\cdot 1}{2\cdot 4}\cdot\frac{h^3}{a^3}-\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\cdot\frac{h^5}{a^5}+\cdots\right],$$

according as a is small or large compared with h, and considering that the mean surface density of the earth is 3 times and the mean density of the whole earth 5.5 times the density of sea water, obtain Siemens's expression, $\frac{6hg}{11r}$, for the diminution of gravity at a point on the ocean where the depth is h. Is the intensity of gravity at the centre of the mouth of a vertical mine shaft 20 feet in diameter appreciably less than before the shaft was dug? Show that if $h=a$, the attraction due to a cylinder of revolution, at the centre of one of its ends, is $2\pi k\rho a(2-\sqrt{2})$. The attraction due to the earth at a point P at a height h above the surface, is $\frac{gr^2}{(r+h)^2}$, or $g\left(1-\frac{2h}{r}\right)$ approximately, where r is the radius of the earth. If ρ_0 is the earth's mean density, $g=\frac{4}{3}\pi k\rho_0 r$. If P is at the centre of a wide plateau of height h made of matter of density ρ, the additional attraction due to the plateau is about $2\pi k\rho h$, or $3g\rho h/2\rho_0 r$, so that if $\rho = \frac{1}{2}\rho_0$, the whole attraction is nearly $g\left(1 - \frac{5h}{4r}\right)$.

31. A vertical solid cylinder of height a and radius r is divided into two parts by a plane through the axis. Show that the resultant horizontal attraction of either part at the centre of the base is

$$2\rho a \cdot \log\left(\frac{r + \sqrt{a^2 + r^2}}{a}\right).$$

32. A right circular cylinder is of infinite length in one direction and is homogeneous. Prove that if the finite extremity be cut off perpendicularly to the generators, the attraction on a unit particle placed at the centre of this end is $\frac{2Mk}{a}$, where M is the mass per unit of length. If the cylinder be elliptic, of the same density and mass per unit of length as before, and of eccentricity e, then the attraction will be n times the former value, where

$$n = \frac{2}{\pi}(1 - e^2)^{\frac{1}{4}} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - e^2 \sin^2\theta}}.$$

33. A homogeneous, right circular cylinder of density ρ stands on the plane $z = 0$, and is infinite in the positive direction of the axis of z. Show that the z component of its attraction at a point P of its base is $k\rho l$, where l is the perimeter of an ellipse having the base for the auxiliary circle and P for one focus.

34. Show that the attraction at any outside point P, due to a uniform plane lamina of any shape, yields a component normal to the lamina, equal to the product of the solid angle subtended at P by the lamina, and a quantity which does not depend upon P's position.

35. Show that the component perpendicular to its axis, of the attraction of a thin, homogeneous, circular, cylindrical sheet of height $2h$ and radius a, has at any point on one of the circular bounding edges of the cylinder the value

$$\frac{\frac{1}{2}M}{\pi a\sqrt{a^2+h^2}}\int_0^{\frac{\pi}{2}}\frac{d\psi}{\sqrt{1-\kappa^2\sin^2\psi}},\text{ where }\kappa^2=\frac{a^2}{a^2+h^2}.$$

36. An infinitely long plane sheet of constant width has a small thickness δ and is made of homogeneous matter of density ρ. This strip cuts a plane perpendicular to its long edges in the line AB: show that the attraction of the strip at any point P in this plane has a component $2k\rho\delta\log(PB/PA)$ parallel to AB, and a component $2k\rho\delta\cdot\angle APB$ perpendicular to AB.

37. Every diameter of a certain circle subtends a plane angle 2θ at a certain point P on the axis of the circle; show that the circle subtends at P the solid angle $2\pi(1-\cos\theta)$.

38. Compare the attractions, at the vertex of a homogeneous oblique cone which has a plane base, due to the whole cone and to so much of it as lies between the vertex and a plane which bisects at right angles the perpendicular drawn from the vertex to the base.

39. Prove the truth of the theorem which Newton states in the following words: "Si corporis attracti, ubi attrahenti contiguum est, attractio longe fortior est, quam cum vel minimo intervallo separantur ab invicem: vires particularum trahentis in recessu corporis attracti, decrescunt in ratione plusquam duplicata distantiarum a particulis. Si particularum, ex quibus corpus attractivum componitur, vires in recessu corporis attracti decrescunt in triplicata, vel plusquam triplicata ratione distantiarum a particulis, attractio longe fortior erit in contactu, quam cum trahens et attractum intervallo vel minimo separantur ab invicem." [*Phil. Nat. Princ. Math.*, Sectio XIII.]

40. Two homogeneous solids made of the same material are bounded by similar surfaces. Show that the intensities of their attractions at two points similarly situated respectively with regard to them, are in the ratio of the corresponding linear dimensions of the solids. Hence prove that the attractions at points on a given diameter inside a solid homogeneous ellipsoid are proportional to the distances of these points from the centre.

41. Prove that the attraction, at very distant points, of any system which has an axis of symmetry, may be represented as emanating from two equal poles of the same sign situated on the axis.

42. Show that the component, at the origin, in the direction of the x axis, of a given particle m, is the same wherever on the surface $m \cdot \cos(x, r) / r^2 = c$, where c is a given constant, the particle lies. If it is anywhere without the surface, the component will be less than if it were anywhere within. Hence prove that the attraction of a given mass M for a point on its surface will be greatest if the boundary of M, referred to the given point, is a surface of the family $\cos\theta = \lambda \cdot r^2$.

43. If the earth be considered as a homogeneous sphere of radius r, and if the force of gravity at its surface be g, show that from a point without the earth, at which the attraction is $\frac{n-1}{n}g$, the area $2\pi r^2\left(1 - \sqrt{\frac{n-1}{n}}\right)$ on the surface of the earth will be visible.

44. The laws of attraction for which the attraction of a homogeneous shell on any external particle is the same as if the shell were concentrated at its centre, are the "law of the inverse square" and the "law of the direct distance."

45. Whatever may be the law of attraction, the intensity of the force exerted by the smaller of two concentric solid homogeneous spheres at any point on the surface of the larger, is to the intensity of the force exerted by the larger at any

point on the surface of the smaller, in the ratio of the square of the radius of the smaller to the square of the radius of the larger. [Minchin.]

46. Prove that if I be an external point and C the centre of a sphere, the sphere on IC as diameter, the sphere with centre I and radius IC, or the polar plane of I, will divide the sphere into two parts which exert equal attractions at I, according as the law of attraction is the inverse square, the inverse cube, or the inverse fourth power of the distance. [St. John's College.]

47. Two sectors are cut from a homogeneous shell bounded by two concentric spherical surfaces of radii r_1 and r_2, by a conical surface of revolution of half angle θ and with vertex at the centre O of the shell. The attractions at a point P without the shell on the axis of the cone, on its inner side, at a distance c from O, due to the portions of the shell which lie respectively without and within the cone are F_1 and F_2. Show that F_1 is equal to the difference between the values when $r = r_2$ and $r = r_1$ of a quantity A, and that F_2 is equal to the difference between the corresponding values of a quantity B where

$$A = \frac{2k\pi\rho}{c^2}\left[\tfrac{1}{3}r^3 - \omega^{\frac{1}{2}}\left(\tfrac{1}{3}r^2 - \tfrac{2}{3}c^2 + c^2\cos^2\theta + \tfrac{1}{3}rc\cos\theta\right)\right.$$
$$\left. + c^3\cos\theta\sin^2\theta\cdot\log\left(\omega^{\frac{1}{2}} + r - c\cos\theta\right)\right],$$
$$B_2 = \frac{2k\pi\rho}{c^2}\left[\tfrac{1}{3}r^3 + \omega^{\frac{1}{2}}\left(\tfrac{1}{3}r^2 - \tfrac{2}{3}c^2 + c^2\cos^2\theta + \tfrac{1}{3}rc\cos\theta\right)\right.$$
$$\left. - c^3\cos\theta\sin^2\theta\cdot\log\left(\omega^{\frac{1}{2}} + r - c\cos\theta\right)\right],$$

and $\omega \equiv c^2 + r^2 - 2\,cr\cos\theta$.

The attractions of the halves of the shell farthest from P and nearest to it are

$$\frac{2k\pi\rho}{3c^2}\left[(r_2^3 - r_1^3) - (r_2^2 - 2c^2)\sqrt{r_2^2 + c^2}\right.$$
$$\left. + (r_1^2 - 2c^2)\sqrt{r_1^2 + c^2}\right],$$

and
$$\frac{2\,k\pi\rho}{3\,c^2}\left[(r_2^3 - r_1^3) + (r_2^2 - 2\,c^2)\sqrt{r_2^2 + c^2} - (r_1^2 - 2\,c^2)\sqrt{r_1^2 + c^2}\right]$$

respectively. If the mass of the *whole* shell is M and if the shell is thin, the attractions at P due to the sectors are

$$\frac{kM}{2\,c^2}\left(1 - \frac{r - c\cos\theta}{PL}\right) \text{ and } \frac{kM}{2\,c^2}\left(1 + \frac{r - c\cos\theta}{PL}\right),$$

where L is any point on the common rim of the sectors.

48. Prove that the attraction due to a homogeneous hemisphere of radius r is zero at a point in the axis of the hemisphere distant $\frac{3}{7}\,r$ approximately from the centre of the base.

49. A segment of height h, cut from a homogeneous sphere of density ρ and radius a by a plane distant $a - h$ from the centre of the sphere, attracts a unit particle on the axis of the segment at a distance b, greater than the radius, from the centre of the sphere, with a force

$$2\pi k\rho\left[h + \frac{1}{3\,(c + a)^2}\left\{(2\,c^2 + 3\,ac)\,c - (2\,c^2 + 3\,ac + ah + ch)\sqrt{c^2 + 2\,ch + 2\,ah}\right\}\right], \text{ where } c = b - a.$$

If $c = 0$, this becomes $2\,\pi k\rho h\left\{1 - \frac{1}{3}\sqrt{\frac{2\,h}{a}}\right\}\cdot$ Assuming this to be true, show that the attraction of a homogeneous hemisphere upon a particle at its vertex is to the attraction of the circumscribing cylinder of the same density as 529 to 586, nearly. Show that the attraction, at its vertex, of a slice 2 miles thick cut from the earth, and the attraction of an infinite disc of the same thickness and density upon a point at the centre of one of its faces, differ by about one per cent of either.

50. Show that if the earth were made up of two homogeneous solid hemispheres of densities ρ and ρ' separated by the plane

of the equator, the deviation of the plumb line from the zenith at any point of the equator would be $\tan^{-1}\left(\frac{2}{\pi}\cdot\frac{\rho-\rho'}{\rho+\rho'}\right)$.

51. Show that the attraction at the origin due to the homogeneous solid bounded by the surface obtained by revolving one loop of the curve $r^2 = a^2\cdot\cos 2\,\theta$, is $\frac{1}{3}\,\pi akp$.

52. A mountain of the form of a surface of revolution with vertical axis and elliptic outline stands on a horizontal plane which contains the centre of the ellipse. Find the horizontal component of its attraction at a point of the base. Show that if the mountain is 2 miles high and 4 miles broad at the base, and if the density of the mountain and of all the matter in its neighborhood is half the mean density of the earth, the plumb lines close to its base on the north and south sides will make with each other an angle greater by about 51 seconds of arc than the corresponding difference of geocentric latitude.

53. The attraction at the point $(0, 0, -b)$ of so much of the homogeneous paraboloid $x^2 + y^2 = \lambda z$ as lies between the planes $z = 0$, $z = h$ is

$$2\,\pi k\rho\,\{h - \sqrt{(b+h)^2 + h\lambda} + b - \tfrac{1}{2}\,\lambda\cdot\log(2\,b + \tfrac{1}{2}\,\lambda)$$
$$+ \tfrac{1}{2}\,\lambda\cdot\log(\sqrt{(b+h)^2 + h\lambda} + b + h + \tfrac{1}{2}\,\lambda)\}.$$

54. If a body M be divided into two rigid portions, A and B, the resultant action of each portion upon itself is nil, and the attraction between A and B is the same mathematically as the attraction between M and B. To find, therefore, the attraction between two equal homogeneous hemispheres so placed as to form a sphere, we may integrate through either hemisphere the product of the density and the component normal to the flat face of the hemisphere, of the attraction due to the whole sphere. Show that the result is $3\,kM^2/16\,a^2$.

55. Show that the resultant attraction between the two parts into which a homogeneous sphere is divided by a plane

is equal to the mass of either part multiplied by the intensity of gravitation at its centre of mass.

56. Prove that the pressure per unit of length on any normal section of a spherical shell of mass M and radius a, due to the mutual gravitation of the particles, tends to the limit $kM^2/16\,\pi a^3$, as the thickness of the shell is indefinitely diminished. [M. T.]

The mass of the unit length of an infinite homogeneous cylinder of revolution of radius a which is divided into two parts by a plane through its axis is M. Show that the pressure between the two parts due to their mutual attractions is $4\,kM^2/3\,\pi a$ per unit length of the cylinder.

57. If R and S denote the components of attraction of a gravitating system symmetrical with respect to a straight axis, taken along and perpendicular to the axis, then

$$D_z R + D_r S + S/r = 0,$$

where r and z are columnar coördinates. [St. John's College.]

58. If the point of application of a force F move by the path s from the point A to the point B, the force is said to *do work* during the journey, equal in amount to the line integral taken along s of the tangential component of F. If the components of F parallel to the coördinate axes are X, Y, Z, and if dx, dy, dz are the projections on these axes of an element ds of the path, we have the expressions

$$W = \int_A^B F\cdot\cos(s, F)\,ds$$

$$= \int_A^B F\,[\cos(x, s)\cdot\cos(x, F)$$

$$+ \cos(y, s)\cdot\cos(y, F) + \cos(z, s)\cdot\cos(z, F)]\,ds$$

$$= \int_A^B [X\cdot\cos(x, s) + Y\cdot\cos(y, s) + Z\cdot\cos(z, s)]\,ds$$

$$= \int_A^B X\,dx + Y\,dy + Z\,dz.$$

If a function Ω exists such that

$$X = D_x\Omega,\ Y = D_y\Omega,\ Z = D_z\Omega\ ;\ \ W = \int_A^B d\Omega = \Omega_B - \Omega_A :$$

such a function is called a *potential function* or a *force function* of the given force. The work done by a force which has a potential function, when its point of application moves completely around any closed path, is zero, and such a force is said to be *conservative.* The work done by a conservative force as its point of application moves from A to B is independent of the path s.

Prove by actual integration along the different paths, that the work done by the force $X = 3x^2 + 2y,\ Y = 4y^3 + 2x,\ Z = 0$, when its point of application moves from the origin to the point (2, 2, 0), is 32, whether the path be a straight line, or the parabola $y^2 = 2x$ in the xy plane, or a combination of a straight line from the origin to (2, 0, 0) and another straight line from this point to (2, 2, 0). Show that the derivative with respect to x of any function of the form $x^3 + 2xy + f(y)$, where f is arbitrary, will yield X, and that, by a proper choice of f, the derivative with respect to y can be made equal to Y; so that a force function exists. Prove by actual integration along the paths that the work done by the force

$$X = 3x^2 + 2y,\ Y = 4y^3 + x,\ Z = 0,$$

as its point of application moves from the origin to (2, 2, 0), is not independent of the path. In this case no potential function exists, since it is impossible to give such a form to f, in the general expression $[x^3 + 2xy + f(y)]$, which has X for its partial derivative with respect to x, that the partial derivative of the expression with respect to y shall be Y.

Since the order of successive partial differentiations of any analytic function is immaterial,

$$D_xD_y\Omega = D_yD_x\Omega,\ D_xD_z\Omega = D_zD_x\Omega,\ D_yD_z\Omega = D_zD_y\Omega$$

or $$D_yZ = D_zY,\ D_zX = D_xZ,\ D_xY = D_yX.$$

Show that this necessary condition for the existence of a force function is also a sufficient one.

59. Prove that if we have matter attracted to any number of fixed centres with forces proportional to any function of the distance, or if we have matter every particle of which attracts every other particle according to any function of the distance between the particles, there exists a potential function the derivative of which in any direction at any point gives the intensity of the force which would solicit a unit quantity of matter concentrated at the point to move in the given direction.

60. If r represents the distance of any point Q on a surface S from a fixed point P, and if α is the angle between PQ and the normal to S at Q, drawn always from the same side of the surface, $\int\frac{\cos\alpha}{r^2}\,dS$, taken over any portion of the surface, gives in absolute value the solid angle subtended at P by this portion, and, in the case of a closed surface, this value is 4π, 2π, or 0, according as P is within, on, or without S.

Prove that the volume of the solid enclosed by any surface S is the absolute value of $\frac{1}{3}\int r\cos\alpha\,dS$ taken over the surface, whether P is within or without S. Show that it is possible to find an analogous expression, $\frac{1}{2}\int r\cos\alpha\,ds$, for the area enclosed by a plane curve, and explain in this case the notation.

61. Show that the absolute value of the component parallel to the axis of x, of the force at a point P, within or without a homogeneous solid body of any form, due to the attraction of this body, is $\rho\int\frac{\cos(x,n)\cdot dS}{r}$, where n is an interior normal, taken all over the bounding surface; and prove that the component parallel to the axis of x of the force, at a point P, due to the attraction of a homogeneous infinite cylinder

with generating lines parallel to the axis of z, is of the form $2\mu\int\cos(x, n)\cdot\log r\cdot ds$, where the integral is to be extended around the contour of the section of the cylinder made by a plane through P perpendicular to the axis of z.

62. The space within a closed surface S is filled with homogeneous matter of density ρ. Prove that the value at the point P, of the potential function due to the distribution, is $\frac{1}{2}\rho\int\cos\alpha\, dS$, where α is the angle which the normal to the surface, drawn inward at any point Q on it, makes with QP.

63. Two distributions of gravitating matter possess a common closed equipotential surface. Prove that if all the matter of both distributions be within this surface, the potentials at the surface due to the two distributions are to each other as the masses.

64. Prove that if two different bodies have the same level surfaces throughout any empty space, their potential functions throughout that space are connected by a linear relation. That the level surfaces should be the same, it is only necessary that the resultant forces due to the two bodies should coincide in direction.

65. Show that if two distributions of matter have in common an equipotential surface which surrounds them both, all their equipotential surfaces outside this will be common.

66. Show that if we have matter every particle of which attracts every other particle with a force proportional to the nth power of the distance, the attraction at any point within a quantity of the matter will be infinite if $n+2<0$. [Minchin.]

67. Show that if u, v, and w are any three solutions of Laplace's Equation,

$$\nabla^2(uvw) = u\cdot\nabla^2(vw) + v\cdot\nabla^2(uw) + w\cdot\nabla^2(uv)$$

68. Show that the potential function due to a solid hemisphere of radius a and density ρ, at an external point P situated on the axis at a distance ξ from the centre, is

$$V=\frac{2\,\pi\rho}{3\,\xi}\left\{a^3\pm\left[\,(a^2+\xi^2)^{\frac{3}{2}}-\xi^3-\frac{3}{2}\,a^2\,\xi\,\right]\right\},$$

the upper or lower sign being taken according as P is on the convex or plane side of the body.

69. A sphere with centre at the origin has a radius r and a density given by the law $\rho=ax+by+cz$. Prove that the value at any external point (x, y, z), at a distance R from the origin, of the potential function due to the sphere, is $4\,\pi r^5\,(ax+by+cz)\,/\,15\,R^3$.

70. An infinite cylinder of radius a has a cylindrical cavity of radius b cut out of it. The axes of the cylinders are parallel but not coincident, and the surfaces do not intersect. Show that the equipotential surfaces are cylinders the equations of which are:

(i) $r_a^{\,2}-r_b^{\,2}=C_1$ within the cavity;

(ii) $r_a^{\,2}-2\,b^2\log\dfrac{r_b}{b}=C_2$ within the mass;

(iii) $a^2\log\left(\dfrac{r_a}{a}\right)-b^2\log\left(\dfrac{r_b}{b}\right)=C_3$ in outside space;

where r_a and r_b are the distances from the axes of the cylinder and cavity respectively.

71. From a homogeneous sphere of density ρ and radius a is cut an eccentric spherical cavity of radius b. The distances of any point P from the centre of the sphere and the centre of the cavity are r_1 and r_2 respectively. Show that V_P, the value of the potential function at P, is given by the first, second, or third of the subjoined equations,

$$r_2^{\,2}-r_1^{\,2}=\frac{3}{2\,\pi\rho}\left\{V_P+2\,\pi\rho\,b^2-2\,\pi\rho\,a^2\right\},$$

$$r_1^2 + \frac{2b^3}{r_2} = 3\left(a^2 - \frac{V_P}{2\pi\rho}\right),$$

$$3V_P = 4\pi\rho\left(\frac{a^3}{r_1} - \frac{b^3}{r_2}\right),$$

according as P is within the cavity, within the mass, or without the mass. Indicate by a rough drawing the form of a line of force within the cavity.

72. Show that the lines of force due to a uniform straight rod are hyperbolas which have the ends of the rod for foci.

73. Show that formula [59] might be written

$$V_P = \mu \cdot \log(\operatorname{ctn} \tfrac{1}{2} PBA \cdot \operatorname{ctn} \tfrac{1}{2} PAB).$$

74. A number (n) of equal, infinitely long, homogeneous, straight filaments, all parallel to each other, cut the xy plane normally in points which lie uniformly distributed on a circumference of radius a with centre at the origin. One of these points is at the point $(a, 0)$. Show that the value of the potential function at the point (r, θ) is

$$m \cdot \log(r^{2n} - 2a^n r^n \cos n\theta + a^{2n}).$$

75. If the law of attraction were that of the inverse nth power of the distance, we should have

$$\nabla^2 V = (n-2)\iiint \frac{\rho\, d\tau}{r^{(n+1)}}.$$

If the density had the same sign throughout a distribution of matter, the potential function could not be constant in any region of empty space unless n were equal to 2.

76. In the case of matter every particle of which attracts every other particle with a force proportional to the product of their masses and a function (f) of the distance, we have $\nabla^2 V \equiv \iiint [2f(r)/r + f'(r)]\,\rho\, d\tau$. Show that V cannot satisfy Laplace's Equation unless $f(r) \equiv K/r^2$.

77. If instead of the polar coördinates r, θ, ϕ, the independent variables are r, μ, ϕ, where $\mu \equiv \cos\theta$, Poisson's Equation becomes

$$D_r(r^2 \cdot D_r V) + D_\mu[(1-\mu^2)D_\mu V] + D^2{}_\phi V/(1-\mu^2) = -4\pi\rho r^2.$$

78. If instead of the spherical coördinates r, θ, ϕ, the coördinates u, w, ϕ be used, where $u \equiv 1/r$, and $w \equiv \log\tan\frac{1}{2}\theta$, Laplace's Equation becomes

$$\sin^2(2\tan^{-1}e^w)u^2 \cdot D_u{}^2 V + D_\phi{}^2 V + D_w{}^2 V = 0.$$

79. Show that if matter be distributed symmetrically about an axis, and if $4a$, $4a'$ be the latera recta of the two confocal parabolas, with this line as axis, which meet at any point, Laplace's Equation may be written in the form

$$D_a(aD_a V) + D_{a'}(a'D_{a'}V) = 0.$$

80. Prove that at the surface of an attracting body, $D_x{}^2 V$, $D_y{}^2 V$, $D_z{}^2 V$ are discontinuous in such a manner that if n represents an interior normal drawn to the surface, the values of these quantities at any point just within the attracting mass are smaller than at a neighboring point just without, by the quantities $4\pi\rho\cos^2(x, n)$, $4\pi\rho\cos^2(y, n)$, $4\pi\rho\cos^2(z, n)$, respectively.

81. A portion of a spherical surface is occupied by a thin shell of matter of uniform density σ, which attracts according to the Newtonian Law. Prove that the value, at any point on the remaining portion of the surface, of the potential function due to this distribution of matter, is $a\sigma\omega$, where a is the diameter of the sphere and ω the solid angle subtended at the point by the contour of the portion of the surface occupied by matter.

82. Show that in so far as a transformation from one set of rectangular axes to another is concerned, $D_x{}^2 V + D_y{}^2 V + D_z{}^2 V$ and $(D_x V)^2 + (D_y V)^2 + (D_z V)^2$ are differential invariants.

83. The potential function at all points external to the sphere

$$x^2 + y^2 + z^2 = a^2$$

is

$$a^5 (\alpha x^2 + \beta y^2 + \gamma z^2 + 2\,\alpha' yz + 2\,\beta' xz + 2\,\gamma' xy)\,/\,r^5.$$

Show that if there be no matter in this region, α, β, and γ must satisfy a certain relation. Show that if inside the sphere the density be uniform, the value there of the potential function will be

$$c + \lambda x^2 + \mu y^2 + \nu z^2 + 2\,\alpha' yz + 2\,\beta' zx + 2\,\gamma' xy,$$

where c, λ, μ, and ν are known. Find the condition that under these circumstances the equipotential surfaces inside the sphere should be ellipsoids similar to $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

84. Prove that if $\int_r^\infty \phi(r) \cdot dr \equiv \chi(r)$ and $\int_r^\infty r \cdot \chi(r)\, dr \equiv \psi(r)$, and if ϕ, χ, and ψ vanish at infinity and are finite for finite values of r; $mm'\,\chi(r)$ represents (1) the work done under an attracting force $mm'\,\phi(r)$ in bringing a particle of mass m' from infinity to a point distant r from another mass m; (2) the component, parallel to the rod, of the attraction of a particle m on a straight slender rod of line density m', if the end of the rod is at a distance r from m and the other end at infinity. Show also that $2\,\pi\sigma m \cdot \psi(z)$ represents (1) the work done in bringing from infinity to a point distant z from a thin lamina of surface density σ, a particle of mass m; (2) the attraction of a particle m, placed at a distance z from the plane surface of an infinite solid of constant density σ.

85. Show that if s represents a direction which makes the angles α, β, γ with the coördinate axes,

$$\begin{aligned} D_s^2 V = D_x^2 V \cos^2\alpha + D_y^2 V \cos^2\beta \\ + D_z^2 V \cos^2\gamma + 2\,D_x D_y V \cos\alpha \cos\beta \\ + 2\,D_y D_z V \cos\beta \cos\gamma + 2\,D_x D_z V \cos\gamma \cos\alpha. \end{aligned}$$

86. When the line of action of the attraction of a body at every point of external space passes through a point O fixed in the body, the body is said to be *centrobaric* and O is called the *baric centre.* The lines of force in external space are straight lines passing through O, and the equipotential surfaces are spherical surfaces with centre at the baric centre. Show that the whole external field must under these circumstances be the same as that due to a mass equal to that of the body, concentrated at O. Show that if at internal points also the line of action of the force always passes through O, the density of the body is a function only of the distance from O. The centre of gravity of a finite centrobaric distribution is the baric centre. A distribution cannot be centrobaric unless every axis drawn through its centre of gravity is a principal axis. If for any finite space outside it a body is centrobaric, it must be centrobaric for all the rest of outside space. A distribution which consists of a spherical distribution and a distribution the potential function due to which at all outside points is zero is evidently centrobaric.

87. Show that if O is a fixed origin within or near a distribution M' of attracting or repelling matter, if P' is any point of M' and P any point without M' more distant from O than any point of M' is, and if $P \equiv (x, y, z)$, $P' \equiv (x', y', z')$, $OP \equiv r$, $OP' \equiv r'$, $POP' \equiv \phi$; the value at P of the potential function due to M' is equal to

$$V_P = \iiint \left[1 - \frac{2\,r' \cdot \cos\phi}{r} + \frac{r'^2}{r^2}\right]^{-\frac{1}{2}} \cdot \frac{dm'}{r}$$

$$= \frac{M'}{r} + \frac{1}{r^2}\iiint r' \cdot \cos\phi \cdot dm'$$

$$+ \frac{1}{2\,r^3}\iiint (2\,r'^2 - 3\,r'^2 \cdot \sin^2\phi)\,dm' + \cdots$$

Show that if A, B, C, and I are the moments of inertia of M' about the coördinate axes and about OP respectively,

$$A + B + C = \iiint 2\, r'^2 dm' \text{ and } I = \iiint r'^2 \cdot \sin^2\phi \cdot dm',$$

and that if O is the centre of gravity of M', the second term of the development vanishes so that

$$V_P = M'/r + (A + B + C - 3\,I)/2\,r^3 + \cdots.$$

If M' is centrobaric and if O is the baric centre, V is a function of r only and the coefficients of r in the general development are to be considered as constants.

88. If the law of attraction is expressed by any function, $\phi'(r)$, of the distance, the intensity of the attraction of any homogeneous solid, estimated in a given direction, at any point P, is expressed by the surface integral $\int \phi(r) \cdot \cos\lambda \cdot dS$, where r is the distance from P of any point on the surface bounding the solid, dS the element of this surface, and λ the angle made by the normal to the element with the given direction. [Minchin.]

89. The function $f(x^p y)$ can satisfy Laplace's Equation only if $p = 1$, or -1, or 0.

90. The invariable line which joins the centres (A_0, B_0) of two homogeneous spheres, A and B, moving under their mutual attraction, revolves with uniform angular velocity, ω, about the centre of gravity, C, of the two. One of the spheres, A, does not rotate, but every line in it remains parallel to itself during the revolution. Show that every particle of A moves in a circle of radius equal to the distance of A's centre from C, and is at every instant at the end of a diameter parallel to B_0A_0. Under these circumstances a loose particle at D on A's surface must in general be constrained to keep it moving in its path.

If we denote the radius of A by a, the distances B_0A_0, CA_0 by d and r, and the mass of B by M, the resultant force on a particle of mass m resting on A at D [Fig. 123] has the intensity $m\omega^2 r \equiv kmM/d^2$ and a direction DT parallel to A_0B_0, while the attraction of B upon the particle has the intensity $kMm/\overline{B_0D}^2$ and the direction DB_0. Show that if a is fairly small compared with d, a constraining force equal to $3\,akMm\,(\sin 2\,\theta)/(2\,d^3)$, where $\theta \equiv CAD$, must be exerted on m in a direction perpendicular to A_0D to prevent its sliding on A's surface.

Assuming A to be the earth, of mass M' and radius a, and B, the moon, of mass $\frac{1}{81}\,M'$, with centre distant $60\,a$ from the earth's centre, prove that the maximum horizontal lunar tide-generating force on the earth's surface is to the force of terrestrial gravitation as 1 to 11,500,000, nearly. Find approximately the "vertical tide-generating force" at the points on the earth's surface nearest and farthest from the moon.

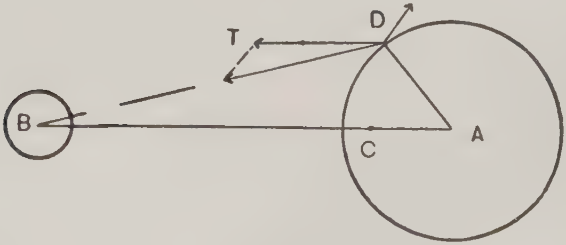

FIG. 123.

[The student is strongly advised to read in this connection Prof. G. H. Darwin's charming Lowell Lectures on the Tides.]

91. Supposing that a sphere of water is brought together by the mutual attractions of its particles from a state of infinite diffusion, and that the amount of work done by these forces is sufficient to raise the temperature of the sphere

1° C. Show that the radius of the sphere is about one-fortieth of the radius of the earth, if the earth's radius be 637×10^6 centimetres, and if one water-gramme-centigrade-degree be equivalent to 4.2×10^7 ergs. [Minchin.]

92. The value at any point (x, y, z) of the potential function due to any system of attracting matter at a finite distance is V, the forces due to the attraction of this matter at any point (x', y', z') is F', the value at this point of the potential function V', and the density ρ'. Show that

$$V^2 = \frac{1}{2\pi}\int\int\int \frac{(4\pi\rho' V' - F'^2)\,dx'dy'dz'}{[(x'-x)^2 + (y'-y)^2 + (z'-z)^2]^{\frac{1}{2}}},$$

where the integration takes in all space.

93. Prove that the rise of sea level in a shallow sea caused by the attraction of a homogeneous hemispherical mountain of radius c rising from it with its base at sea level, is approximately $\rho' c^2/2\rho a$, where ρ' is the density of the mass of the mountain, ρ the mean density of the earth, and a its radius.

94. A fixed gravitating sphere is partly covered by an ocean extending over the northern side of a parallel of colatitude λ. A distant fixed gravitating body M is situated on the north axis of this small circle. Prove that if the self-attraction of the ocean be neglected, M will cause a rise of water at the north pole approximately equal to $\kappa \sin^2 \frac{1}{2}\lambda$, where κ is what the rise would be if the whole sphere were covered.

95. Show that if a finite distribution consists of m units of positive matter and m units of negative matter, anyhow distributed, it is possible to draw, with any given finite point as centre, a spherical surface so large that the whole flow of force through it, *reckoned arithmetically*, shall be as small as we please. Prove that the lines of force are all closed.

96. Imagine any point P in empty space near a distribution of repelling matter to be taken as origin of a system of orthogonal Cartesian coördinates with axis of z coincident with the

normal to the equipotential surface which passes through P. V will then be given by an equation of the form $V = f(x, y, z)$, where $D_x f$, $D_y f$ vanish at P, and $-D_z f$ is the force F in the direction of the z axis. If Q is a point near P on the section of the surface $V = V_P$ made by the xz plane, and if we denote the coördinates of Q by $(\Delta x, 0, \Delta z)$, the radius of curvature at P of this section is $\lim\limits_{\Delta x \doteq 0}\left(\dfrac{\Delta x^2}{2\,\Delta z}\right)$, and Δz is in general of higher order than Δx.

$$V_Q = V_P + \Delta x \cdot D_x V + \Delta z \cdot D_z V$$
$$+ \tfrac{1}{2}\,\Delta x^2 \cdot D_x^2 V + \text{terms of higher order.}$$

Since $V_Q = V_P$, and $D_x V$ vanishes at P, $D_x^2 V = \dfrac{F}{R_1}\cdot$ Prove similarly that $D_y^2 V = \dfrac{F}{R_2}$ and then, by Laplace's Equation, that

$$D_z^2 V = -F\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

Illustrate these results by an example.

97. If a distribution of active matter is symmetrical about a straight line (the axis of x) and if r represents the distance of any point from this axis, the potential function involves r and x only and the equipotential surfaces are surfaces of revolution. Consider one of these surfaces, S_0, on which V has the value V_0, and let the "flux of force" through so much of S_0 as lies between some fixed plane ($x = x_0$) perpendicular to the x axis, and the plane $x = x$, be represented by the function $2\,\pi\mu$, then if ds is the element of the generating curve of S_0 between x and $x + \Delta x$, and if r is the distance of ds from the x axis, the area of the strip of S_0 between x and $x + \Delta x$ is approximately $2\,\pi r \cdot ds$, the flux of force through it is $-2\,\pi r \cdot D_n V \cdot ds$, and this flux is the change made in $2\,\pi\mu$ by increasing x by Δx. We may write, therefore, $D_s \mu = -r \cdot D_n V$,

and, if α is the angle which the exterior normal to ds makes with the x axis,

$$D_s\mu \equiv D_x\mu \cdot \sin\alpha - D_r\mu \cdot \cos\alpha, \;\; D_n V \equiv D_x V \cdot \cos\alpha + D_r V \cdot \sin\alpha,$$

and the equation becomes

$$\sin\alpha\,(D_x\mu - r \cdot D_r V) - \cos\alpha\,(D_r\mu + r \cdot D_x V) = 0.$$

If this equation is to hold everywhere on every equipotential surface, the coefficients of $\sin\alpha$ and $\cos\alpha$ must vanish and μ is determined (apart from an additive constant to be chosen at pleasure) by the equations $D_x\mu = r \cdot D_r V$, $D_r\mu = - r \cdot D_x V$.

Show that the values of μ corresponding to the three familiar potential functions $- X_0 x$, $Mx/(r^2 + x^2)^{\frac{3}{2}}$, $M/(r^2 + x^2)^{\frac{1}{2}}$ are $\frac{1}{2} X_0 r^2$, $Mr^2/(r^2 + x^2)^{\frac{3}{2}}$, and $- Mx/(r^2 + x^2)^{\frac{1}{2}}$. Discuss the physical meanings of these results.

The function μ defined above is sometimes called "Stokes's Flux Function." It is clear that the level surfaces of the functions V and μ, both of which are symmetrical about the x axis, cut each other orthogonally and that the generating line of any level surface of μ is a line of force. Although any function of μ equated to a constant would serve to represent the forms of analytic lines of force, a special advantage arises from the use of μ itself from the fact that if μ_1 and μ_2 are flux functions corresponding to two different potential functions, V_1 and V_2, due to two distributions of matter, M_1 and M_2, symmetrical about the x axis, $\mu_1 + \mu_2$ is a flux function of $V_1 + V_2$, the potential function due to M_1 and M_2 existing together. If generating lines of the μ_1 surfaces be drawn in a plane, for the numerical values a, $a + \delta$, $a + 2\,\delta$, $a + 3\,\delta$, $a + 4\,\delta$, etc., and the lines of the μ_2 surfaces for the values b, $b + \delta$, $b + 2\,\delta$, $b + 3\,\delta$, $b + 4\,\delta$, etc., δ being any convenient interval, the intersections of the curves $\mu_1 = a + n\delta$, $\mu_2 = b + (m - n)\,\delta$ will be points on the generating lines of the surface $\mu_1 + \mu_2 = a + b + m\delta$. If, then, we fix m for a moment and give to n in succession different integral values, we may get points enough to enable us to draw the line $\mu_1 + \mu_2 = a + b + m\delta$ with sufficient accuracy. This graphical method of drawing lines of force (or equipotential surfaces) has proved in the hands of Maxwell and others extremely fruitful. Draw accurately several of the lines of force due to a charge 20 and a charge -10 concentrated at points 4 inches apart.

98. (*a*) Show that if P, P' are any definite pair of inverse points distant respectively r and r' from the centre O of a spherical surface S of radius a, the ratio $PQ/P'Q$ is equal to the constant a/r' wherever on S the point Q may be. Hence show that if V is the potential function due to a heterogeneous surface distribution on S,

$$V_{P'} = V_P(a/r') \text{ and } D_{r'}(V_{P'}) = -a^3 \cdot D_r V_P \cdot / r'^3 - aV/r'^2.$$

(*b*) Prove that $r^{3/2} \cdot D_r V_P + r'^{3/2} \cdot D_{r'} V_{P'} = -(a V_P \cdot V_{P'})^{1/2}$.
[Routh.]

(*c*) Prove that as both r and r' are made to approach a, limit $(D_r V_P + D_{r'} \cdot V_{P'}) = -V/a$. [Stokes.]

99. If P, P' are any definite pair of inverse points with respect to a right section of an infinitely long cylindrical surface of revolution, and if Q be any variable point on the circumference, $P'Q/PQ$ is equal to the constant r'/a. Show that if the cylinder be covered with a superficial distribution the density of which varies from filament to filament of the surface, $V_{P'} - V_P = 2\log(r'/a) \cdot M$, where M is the amount of matter on the unit length of the surface.

100. V is the potential function due to a volume distribution of density ρ in the region T and a surface distribution of density σ on the surface S. V' is the potential function due to a volume distribution of density ρ' in the region T' and a surface distribution of density σ' on the surface S'. Using all space as the field of Green's Theorem, apply [145] to these functions and interpret the resulting equation as giving an expression for the mutual energy of the distributions.

101. If two systems of matter (M and M'), both shut in by a closed surface S, give rise to potential functions (V and V'), which have equal values at every point of S, whether or not S is an equipotential surface of either system, then V cannot differ from V' at any point outside S, and the algebraic sum of the matter of either system is equal to that of the other.

102. Prove that the level lines of the function $u \equiv F_2(x, y, z)$ on the surface $F_1(x, y, z) = 0$ have direction cosines which are to each other as

$$(D_yF_1 \cdot D_zF_2 - D_zF_1 \cdot D_yF_2),$$

$$(D_zF_1 \cdot D_xF_2 - D_xF_1 \cdot D_zF_2),$$

and $$(D_xF_1 \cdot D_yF_2 - D_yF_1 \cdot D_xF_2);$$

and that if these quantities be represented by λ, μ, and ν, respectively, the direction cosines, at the point (x, y, z), of a curve which lies on F_1 and cuts orthogonally at that point a level line of u on the surface, are to each other as

$$(\mu \cdot D_zF_1 - \nu \cdot D_yF_1) : (\nu \cdot D_xF_1 - \lambda \cdot D_zF_1) : (\lambda \cdot D_yF_1 - \mu \cdot D_xF_1).$$

In particular, if $u \equiv x / z$ and if $F_1(x, y, z) \equiv x^2(b^2 - y^2) - a^2z^2$, the level lines of u on F_1 are straight lines, the direction cosines of which at any point P are in the ratio $u_P : 0 : 1$, and since the sum of the squares of these cosines must be equal to unity, the cosines themselves are $u / \sqrt{u^2 + 1}$, 0, and $1 / \sqrt{u^2 + 1}$.

103. In the case of a columnar distribution the density of which varies only with the distance r from a fixed axis, the lines of force are straight lines radiating from the axis (Section 34), and the potential function V and the resultant force D_rV are functions of r alone. If we apply Gauss's Theorem to a cylindrical surface of revolution S, coaxial with the distribution, we learn that $2\pi r D_rV = 4\pi$ times the mass M of the unit length of so much of the distribution as is enclosed by S.

Show that if the distribution is a solid homogeneous repelling cylinder of radius a and density ρ, $D_r V = 2\pi\rho r$ and $V = \pi\rho[r^2 - a^2 + 2a^2 \log a]$, if r is less than a. If r is greater than a, $D_r V = 2\pi\rho a^2/r$ and $V = 2\pi\rho a^2 \log r$. Show also that if the distribution is merely a surface charge of density σ on a cylindrical surface of radius a, $V = 4\pi a\sigma \log a$ within the cylinder, and $V = 4\pi a\sigma \log r$ without.

104. If V is the gravitational potential function belonging to a given distribution M of attracting matter, and if k is the constant of gravitation, the force of gravitation at any point in any direction s, measured in dynes, is the value at that point of $k \cdot D_s V$ or $D_s V_0$, where $V_0 \equiv kV$; and $\nabla^2 V_0 = -4\pi k\rho$. Prove that if M be made to rotate about the axis of z with constant angular velocity ω, if $\Omega \equiv \frac{1}{2}\omega^2(x^2 + y^2)$, and if $V' \equiv V_0 + \Omega$, the apparent force at any point in any direction s is $D_s V'$ and $\nabla^2 V' = -4\pi k\rho + 2\omega^2$. Prove also that if S represents the surface of M, and n a normal to the surface drawn inwards, if v is the volume of the distribution, and if ρ_0 is its mean density,

$$-4\pi k v\rho_0 + 2\omega^2 v = -\int\int D_n V' dS.$$

[R. S. Woodward, "The Gravitational Constant and the Mean Density of the Earth," *Astronomical Journal*, 1898.]

If M represents the earth, a the semiaxis major, and e the eccentricity of the generating ellipse of the earth's spheroid, ϕ and λ the latitude and longitude of dS, we have

$$v = \tfrac{4}{3}\pi a^3 \sqrt{1 - e^2},\ dS = a^2(1 - e^2)\cos\phi\, d\phi\, d\lambda / (1 - e^2 \sin^2\phi)^2,$$

and the acceleration $D_n V'$ is given in centimetres per second per second by the equation $D_n V' = \alpha + \beta \sin^2\phi + \gamma \cos^2\phi \cdot \cos 2\lambda$, where $\alpha = 978$, $\beta = 5.19$, and γ is a constant, so that

$$\int\int D_n V' \cdot dS = \alpha S + \beta \int\int \sin^2\phi\, dS.$$

Show that if $x \equiv e \sin \phi$,

$$S = \frac{2a^2(1-e^2)}{e} \int_0^e \frac{dx}{(1-x^2)^2} \int_0^{2\pi} d\lambda = 4\pi a^2 \left(1 - \tfrac{1}{3} e^2 - \tfrac{1}{15} e^4 - \cdots\right),$$

and

$$\iint \sin^2 \phi \, dS = \frac{2a^2(1-e^2)}{e^3} \int_0^e \frac{x^2 dx}{(1-x^2)^2} \int_0^{2\pi} d\lambda$$

$$= 4\pi a^2 \left(\tfrac{1}{3} + \tfrac{1}{15} e^2 + \tfrac{1}{35} e^4 + \cdots\right),$$

and, assuming that $\log e^2 = \bar{3}.83050$, $\log a = 8.80470$, obtain Professor Woodward's equation, $k\rho_0 = 36797 \times 10^{-11}$.

For a discussion of the value of ρ_0, see Prof. J. H. Poynting's Adams Prize Essay on "The Mean Density of the Earth."

105. If u is single-valued and harmonic at all points of a region but one (the exceptional point being an interior point P), and if u becomes infinite for all paths along which the point (x, y) approaches P, then u can be written in the form $u = a \cdot \log r + v$, where v is single-valued and harmonic at all points of the region. [Bôcher.]

106. If the superficial density of a mass distributed on a spherical surface is inversely proportional to the cube of the distance from a fixed point A, the distribution is centrobaric. If A is inside the surface, it is the baric centre; if A is outside, its inverse point is the baric centre.

107. If the superficial density of a columnar distribution on a cylindrical surface of revolution varies inversely as the square of the distance from a given line parallel to the axis of the cylinder, there is a baric line within the distribution parallel to the axis. Where is this line?

108. A certain distribution M has two mutually exclusive closed equipotential surfaces, S_1 and S_2, upon which V has the different constant values C_1 and C_2. Will the potential function in space without S_1 and S_2 be the same for the original distribution and for distributions on S_1 and S_2 of surface

density $\sigma = -D_n V/4\pi$ together with so much of M as lies without the surfaces?

109. The straight-line tangents at a point to a tube of force which ends there, evidently form a cone of definite solid angle. A number of points, P_1, P_2, P_3, etc., have charges, m_1, m_2, m_3, etc. Show that if at any one of these points there end two tubes the solid angles of the cones of which are ω and ω', the flow of force in the one tube is to the flow of force in the other as $\omega : \omega'$. Show also that if a tube starts with solid angle ω_k at a point P_k where the charge is m_k, and ends with solid angle ω_r at a point P_r where the charge is m_r, $\omega_k m_k$ is numerically equal to $\omega_r m_r$.

110. All the masses of a certain distribution lie within two closed surfaces S_1 and S_2, which exclude each other and are equipotential. All the lines of force which abut on a continuous portion A of S_1 also abut on S_2. All the lines of force which abut on S_1 outside of A are open, while none of the lines which abut on S_2 are open. Show that one of the equipotential surfaces is made up of two lobes, one of which includes S_2 alone and the other both S_1 and S_2. Separating the closed lines of force from the open ones is a surface which passes through the point where the lobes of the surface just mentioned are connected. All the equipotential surfaces are closed.

111. The potential function due to a certain distribution of matter has a value at any point Q which depends only upon the distance, r, of Q from a fixed point O. This value is

$$2\pi(2c + b^2 - a^2), \qquad \tfrac{2}{3}\pi\left(6c + 3b^2 - r^2 - \frac{2a^3}{r}\right),$$

$$4\pi\left(c + \frac{b^3 - a^3}{3r}\right), \quad \text{or} \quad 4\pi\left(\frac{3c^2 + b^3 - a^3}{3r}\right),$$

according as $0 < r < a$, $a < r < b$, $b < r < c$, or $r > c$. What is the distribution? [$\rho = 0$, $\rho = 1$, $\rho = 0$, $\sigma = 1$, $\rho = 0$.]

112. The lines of force due to two similar, homogeneous, infinitely long, straight filaments of repelling matter, parallel to the z axis and cutting the xy plane at the points $(a, 0)$, $(-a, 0)$ are hyperbolas of the family $x^2 - 2\lambda xy - y^2 = a^2$.

113. (*a*) Prove that when there is symmetry about the axis from which θ is measured, $r^m \cdot P_m(\cos\theta)$ and $\dfrac{P_m(\cos\theta)}{r^{m+1}}$, where $P_m(\cos\theta)$ is the coefficient of α^m in the development, in ascending powers of α, of $(1 - 2\alpha\cos\theta + \alpha^2)^{-\frac{1}{2}}$, are particular solutions of Laplace's Equation in polar coördinates; that is, of

$$r \cdot D_r^2(rV) + \frac{1}{\sin\theta} \cdot D_\theta(\sin\theta \cdot D_\theta V) = 0.$$

Hence show that any expression of the form

$$A_0 + A_1 r \cdot P_1(\cos\theta) + A_2 r^2 \cdot P_2(\cos\theta) + \cdots + A_n r^n \cdot P_n(\cos\theta)$$
$$+ \frac{B_0}{r} + \frac{B_1 \cdot P_1(\cos\theta)}{r^2} + \frac{B_3 \cdot P_2(\cos\theta)}{r^3} + \cdots + \frac{B_m \cdot P_m(\cos\theta)}{r^{m+1}},$$

where A_0, B_0, A_1, B_1, A_2, B_2, etc., are arbitrary constants, satisfies the equation. The P's here introduced are sometimes called Legendre's Coefficients, sometimes Zonal Surface Spherical Harmonics.

(*b*) Show that

$$\begin{aligned} P_0(\mu) &= 1, \\ P_1(\mu) &= \mu, \\ P_2(\mu) &= \tfrac{1}{2}(3\mu^2 - 1), \\ P_3(\mu) &= \tfrac{1}{2}(5\mu^3 - 3\mu), \\ P_4(\mu) &= \tfrac{1}{8}(35\mu^4 - 30\mu^2 + 3), \\ P_5(\mu) &= \tfrac{1}{8}(63\mu^5 - 70\mu^3 + 15\mu). \end{aligned}$$

(*c*) Show that when $\theta = 0$, $P_m(\cos\theta)$ or $P_m(1)$, the coefficient of α^m in the development of $(1 - \alpha)^{-1}$, is equal to 1.

114. Prove that if in any case of symmetry about a line, a convergent series $a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$ represents the

value of the potential function at a point Q distant z from the fixed point O, both O and Q being on the line; then the series

$$a_0 P_0(\cos\theta) + a_1 r P_1(\cos\theta) + a_2 r^2 P_2(\cos\theta) + a_3 r^3 P_3(\cos\theta) + \cdots,$$

formed by writing instead of z^m in the former series $r^m \cdot P_m(\cos\theta)$, will represent in polar coördinates with O as origin and the given line as axis from which θ is measured, a finite, single-valued function which satisfies Laplace's Equation and for all points on the given line on the positive side of O, where $\theta = 0$ and $r = z$, has the same values as the given series. Given the radius of convergence of the first series, within what limits can we safely use the second series? If any portion of the given line traverses a region of empty space, does the new series represent the potential function at all points in this region within the limits of convergency of the series?

115. Prove that if in any case of symmetry about a line, a convergent series $\frac{a_1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \cdots$ represents the value of the potential function, the series

$$\frac{a_1 P_0(\cos\theta)}{r} + \frac{a_2 P_1(\cos\theta)}{r^2} + \frac{a_3 P_2(\cos\theta)}{r^3} + \cdots,$$

formed by writing instead of $\frac{1}{z^{m+1}}$ in the former series, $\frac{P_m(\cos\theta)}{r^{m+1}}$, will represent, so long as the new series is convergent, a finite, single-valued function which satisfies Laplace's Equation and, for all points on the line on the positive side of O, has the same values as the given series.

116. Prove that the potential function due to a uniform circular ring of mass M, of radius a, and of small cross-section, is equal to

$$\frac{M}{r}\left(1 - \frac{1}{2}\cdot\frac{a^2 \cdot P_2(\cos\theta)}{r^2} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{a^4 \cdot P_4(\cos\theta)}{r^4} - \cdots\right)$$

if $a < r$, and equal to

$$\frac{M}{a}\left(1 - \frac{1}{2}\cdot\frac{r^2 \cdot P_2(\cos\theta)}{a^2} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{r^4 \cdot P_4(\cos\theta)}{a^4} - \cdots\right)$$

if $a > r$, where the centre of the ring is the origin, and the axis of the ring the axis from which θ is measured.

117. Prove that the potential function due to a uniform circular disc of mass M, of radius a, and of small thickness, is equal to

$$\frac{2M}{a^2}\left(\frac{1}{2}\cdot\frac{a^2}{r}-\frac{1}{2^2\cdot 2!}\cdot\frac{a^4\cdot P_2(\cos\theta)}{r^3}+\frac{1\cdot 3}{2^3\cdot 3!}\cdot\frac{a^6\cdot P_4(\cos\theta)}{r^5}-\cdots\right)$$

if $a < r$, and to

$$\frac{2M}{a^2}\left(a-r\cdot P_1(\cos\theta)+\frac{1}{2}\cdot\frac{r^2\cdot P_2(\cos\theta)}{a}-\frac{1}{2^2\cdot 2!}\cdot\frac{r^4\cdot P_4(\cos\theta)}{a^3}+\cdots\right)$$

if $a > r$, when the centre of the ring is the origin.

118. Show that the expression $\pm(r^2 - c^2 + y^2)/y$ of equation [21], page 12, is numerically equal to the length, k, of the chord of the sphere, formed by a radius vector drawn from P to a point L on the surface, distant y from P. The sign is to be taken negative or positive, according as L is or is not visible from P. Hence find an expression, $\pi\sigma a(k_1 \pm k_2)/c^2$, for the intensity of the attraction of an "annulus" of a thin spherical shell lying between two parallels of latitude, at any point P on the axis.

119. A thin spherical shell of radius a attracts an internal particle P at a distance c from its centre. If the shell be divided into two parts by a plane through P perpendicular to the radius, the resultant attraction of each part at P is $2\pi\sigma a[a - \sqrt{a^2 - c^2}]/c^2$. [Todhunter's *History of Attraction.*]

120. The equation of the surface of an infinitely long homogeneous cylinder of density ρ, the lines of which are parallel to the z axis, being $r = f(\theta)$, a filament of the cylinder of cross-section $r\,dr\,d\theta$ contributes to the components (X, Y) of the attraction at the origin the amounts $2\rho\cos\theta\cdot dr\cdot d\theta$ and $2\rho\sin\theta\cdot dr\cdot d\theta$ respectively. If the cross-section of the cylinder is an ellipse of semiaxes a and b and if the origin is on the surface and distant y_0, x_0 respectively from the

principal planes, the equation of the surface may be written in the form

$$r = 2(b^2x_0 \cos\theta + a^2y_0 \sin\theta) / (b^2\cos^2\theta + a^2\sin^2\theta).$$

Assuming that

$$\int \frac{dx}{a + b\tan^2 x} = \frac{1}{a-b}\left[x - \sqrt{\frac{b}{a}} \cdot \tan^{-1}\left(\sqrt{\frac{b}{a}} \cdot \tan x\right)\right],$$

prove that in this case

$$X = 4\pi\rho b x_0 / (a+b), \quad Y = 4\pi\rho a y_0 / (a+b)$$

and that the resultant force has the intensity $4M/(a+b)$, where M is the mass of the unit length of the cylinder. Prove also, by a method analogous to that of Section 12, that the attraction due to a homogeneous shell bounded by two concentric, similar, and similarly placed elliptic cylindrical surfaces is zero within the shell, and that the attraction components (X, Y) at any point within a solid homogeneous elliptic cylinder are proportional to x and y respectively.

121. If two confocal ellipses (s and s') have semiaxes (a, b) and (a', b'), a point (x, y) on s is said to *correspond* to a point (x', y') on s', if $x/x' = a/a'$ and $y/y' = b/b'$. Show that if P_1 and P_2 are any two points on s and P_1' and P_2', the corresponding points on s', $P_1P_2' = P_2P_1'$. Hence prove (Section 51) that, if two homogeneous, solid, confocal, elliptic cylinders of the same density be divided into corresponding thin strips by planes parallel to the xz plane, the x component of the attraction of any strip of the first at a point P' on the second, is to the x component of the attraction of the corresponding strip of the second at a point P on the first corresponding to P', as b is to b'. The two components of the attraction of the whole of the first cylinder at P' are to the same components of the attraction of the second cylinder at P, as b to b' and as a to a'.

122. It follows from the results stated in the last two problems, that if the components at an outside point Q' of the attraction due to a solid homogeneous elliptic cylinder of density ρ bounded by the surface s (Fig. 124) be X' and Y', if a surface s' confocal to s be drawn through Q', and if X and Y are the components, at Q on s which corresponds to Q' on s', of the attraction of a cylinder of density ρ bounded by s'; $X'/X = b/b'$, $Y'/Y = a/a'$, where a and b are the semiaxes of s, and a' and b' those of s'. Show that X, Y are the components at Q of the attraction due to a cylinder of density ρ, bounded by a surface s'' drawn through Q, similar to s'. Show also that, if the coördinates of Q are x, y,

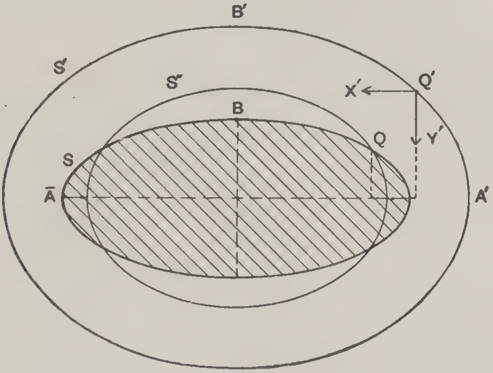

FIG. 124.

$$X' = 4\,\pi\rho b x/(a' + b'), \quad Y' = 4\,\pi\rho a y/(a' + b').$$

Prove that, if $a = 5$, $b = 3$, $x' = 4$, and $y' = \frac{10}{3}$; $x = \frac{10}{3}$, $y = \sqrt{5}$, $a' = 6$, $b' = \sqrt{20}$, so that, approximately, $X' = 12\,\rho$, $Y' = 13 \cdot 42 \cdot \rho$.

123. Two parallel planes, the direction cosines of the normals to which are (l, m, n), touch two confocal ellipsoidal surfaces at the points P_1, P_1' respectively. The semiaxes of

the surfaces are (a, b, c) and $(a + da,\ b + db,\ c + dc)$ where, since they are confocal, $d(a^2) = d(b^2) = d(c^2)$. Show that if p and $p + dp$ are the lengths of the perpendiculars dropped from the origin on the tangent planes, $p^2 = a^2l^2 + b^2m^2 + c^2n^2$, and $p \cdot dp = l^2 \cdot d(a^2) + m^2 \cdot d(b^2) + n^2 \cdot d(c^2) = d(a^2)$, so that dp is inversely proportional to p. If the surfaces bound a homogeneous shell, this is called a *thin focaloid*. Show that the thickness of the shell at the point P differs from dp, if at all, by an infinitesimal of higher order, and that a superficial distribution on an ellipsoid with surface density inversely proportional to p is equivalent to a thin focaloid bounded internally by the surface. The thickness of a thin homœoid at any point is directly proportional to p.

124. Show that if the potential function due to a distribution of matter has the value zero at all points outside the ellipsoid $Lx^2 + My^2 + Nz^2 = 1$ and the value $\mu(1 - Lx^2 - Mx^2 - Nz^2)$ at all inside points, the distribution consists of a homogeneous ellipsoid of density $\mu(L + M + N)/2\pi$ and a superficial stratum on it of surface density $-\mu/2\pi p$, where p is the length of the perpendicular dropped from the origin on the tangent plane. Since the surface distribution is equivalent to a thin focaloid, it is clear that the potential function due to a homogeneous ellipsoid has at outside points the same values as the potential function due to a thin focaloid of the same mass coincident with the surface of the ellipsoid. Prove from this that confocal ellipsoids of equal mass have equal potential functions at points outside both.

125. Two homogeneous, solid, confocal ellipsoids of masses M_1 and M_2 attract any particle outside both with forces which have the same direction and are to each other as M_1 to M_2. [Maclaurin.]

126. Show that it follows from the reasoning on pages 123 and 124 that the components, taken parallel to the axes, of the attraction of a homogeneous ellipsoid S at any point P' on the surface of a homogeneous confocal ellipsoid S' of the

same density, are to the corresponding attraction components due to S' at the point P on S which corresponds to P', as the areas of the principal sections of S and S' perpendicular to these components. [Ivory.]

127. We know from the equations of page 191 that, in the case of a prolate ellipsoid uniformly polarized in the direction of the long axis, the depolarizing force is

$$-4\,\pi A\frac{(1-e^2)}{e^2}\left(\frac{1}{2\,e}\cdot\log\frac{1+e}{1-e}-1\right).$$

Prove that if the ratio of a to b is large, this is nearly equal to $-4\,\pi A(b^2/a^2)[\log(2\,a/b)-1]$, and that when $a/b=4$, this approximate result is in error by about 4 per cent.

Show that if we denote the depolarizing force in an ellipsoid of revolution uniformly polarized in the direction of the x axis by λA, λ has the values 12.57, 6.63, 5.16, 4.19, 2.18, 0.95, 0.25, 0.0054, 0.0016, 0.0004, according as a/b is equal to 0, $\frac{1}{2}$, $\frac{3}{4}$, 1, 2, 4, 10, 100, 200, 400.

128. If the quantity c on page 121 be supposed to increase without limit, the limits of the expressions for X and Y are the force components within a homogeneous elliptic cylinder of semiaxes a and b. Making use of the integral

$$\int\frac{dx}{(x+\alpha)^{\frac{3}{2}}(x+\beta)^{\frac{1}{2}}}=\frac{2\,(x+\beta)^{\frac{1}{2}}}{(\alpha-\beta)\,(x+\alpha)^{\frac{1}{2}}},$$

show that these limits are $4\,\pi b\rho x/(a+b)$ and $4\,\pi a\rho y/(a+b)$. Using the form of integral given on page 124 in the seventh line from the bottom, show that if c be made to increase without limit, the limit of X is $-4\,\pi\rho bx/(a'+b')$ and that the corresponding limit of Y is $-4\,\pi\rho ay/(a'+b')$.

Show that the equipotential surfaces within an infinitely long, solid, homogeneous, elliptic cylinder, the semiaxes of which are a and b, are elliptic cylindrical surfaces, the ratio of the semiaxes of any one of which is $\sqrt{a}/\sqrt{b}$.

129. Using the integrals given on page 190, show that if $a = b > c$ and if $\lambda^2 \equiv (a^2 - c^2)/c^2$, we may write the expressions for the attraction components within a homogeneous oblate ellipsoid of revolution, in the form

$$(-3\,Mx/2\,\lambda^3 c^3)\,[\tan^{-1}\lambda - \lambda/(1+\lambda^2)],$$
$$(-3\,My/2\,\lambda^3 c^3)\,[\tan^{-1}\lambda - \lambda/(1+\lambda^2)],$$
$$(-3\,Mz/\lambda^3 c^3)(\lambda - \tan^{-1}\lambda).$$

130. Show that if in the case of the prolate ellipsoid of revolution where $b = c < a$, we put $\lambda \equiv ea/c$, the components of attraction at the inside point (x, y, z) may be written

$$(3\,Mx/\lambda^3 c^3)\,[\lambda/\sqrt{1+\lambda^2} - \log(\lambda + \sqrt{1+\lambda^2})],$$
$$(3\,My/2\,\lambda^3 c^3)\,[\log(\lambda + \sqrt{1+\lambda^2}) - \lambda\sqrt{1+\lambda^2}],$$
$$(3\,Mz/2\,\lambda^3 c^3)\,[\log(\lambda + \sqrt{1+\lambda^2}) - \lambda\sqrt{1+\lambda^2}].$$

131. If these force components be denoted by X, Y, Z, the quantity $(X/x + Y/y + Z/z)$ is numerically equal to $-4\,\pi\rho$ within any ellipsoid of revolution. This is true in the case of every ellipsoid, as Poisson's Equation shows.

132. If $\alpha \equiv a^2$, $\beta \equiv b^2$, $\gamma \equiv c^2$, and if G_0 has the value given on page 122,

$$\alpha \cdot D_\alpha G_0 + \beta \cdot D_\beta G_0 + \gamma \cdot D\gamma G_0 = -\tfrac{1}{2}\,G_0$$

and

$$2(\alpha - \beta)\,D_\alpha D_\beta G_0 = D_\alpha G_0 - D_\beta G_0.$$

The potential function V satisfies the equation

$$V = \pi\rho abc\,(G_0 + 2\,x^2 \cdot D_\alpha G_0 + 2\,y^2 \cdot D_\beta G_0 + 2\,z^2 \cdot D_\gamma G_0)$$

at every point within a homogeneous ellipsoid.

If (a', b', c') are the semiaxes of an ellipsoidal surface through a point (x, y, z), confocal with the ellipsoid E which has the semiaxes (a, b, c), and if the result of substituting a', b', c' for a, b, c in the expression for G_0 be denoted by G', the value of the potential function due to E at an external point may be written

$$\tfrac{3}{4}\, M \{ G' + 2\, x^2 \cdot D_l G' + 2\, y^2 \cdot D_m G' + 2\, z^2 \cdot D_n G' \}$$

where $$l \equiv a'^2,\ m \equiv b'^2,\ n \equiv c'^2.$$ [Tarleton.]

133. Show that if X, Y, Z are the components of the body forces applied to a mass M of liquid revolving with uniform angular velocity ω about the axis of z, and if p denotes the pressure at the point (x, y, z),

$$dp = (X + \omega^2 x)\, dx + (Y + \omega^2 y)\, dy + Z\, dz,$$

so that at a free surface

$$(X + \omega^2 x)\, dx + (Y + \omega^2 y)\, dy + Z\, dz = 0.$$

134. If the liquid be homogeneous and exposed to its own attraction only, and if the bounding surface be the ellipsoid $b^2c^2x^2 + a^2c^2y^2 + a^2b^2z^2 = a^2b^2c^2$, we have $X = -\tfrac{3}{2} MK_0 x$, $Y = -\tfrac{3}{2} ML_0 y$, $Z = -\tfrac{3}{2} MM_0 z$, and at the free surface

$$b^2c^2xdx + a^2c^2ydy + a^2b^2zdz = 0,$$

so that

$$(\omega^2 - \tfrac{3}{2} MK_0) / b^2c^2 = (\omega^2 - \tfrac{3}{2} ML_0) / a^2c^2 = -\tfrac{3}{2} MM_0 / a^2b^2.$$

Show that this condition is satisfied for a given value of ω by an oblate ellipsoid of revolution (Example 129) for which λ satisfies the equation, $\lambda = \tan[(3\lambda + 2\,\omega^2\lambda^3 / 4\,\pi\rho) / (3 + \lambda^2)]$; but that a prolate ellipsoid of revolution is not a possible form of the bounding surface. [Besant's *Hydromechanics*, Vol. I; Laplace's *Mécanique Céleste*, Vol. III.]

135. Prove that if V be the potential function due to any distribution of matter over a closed surface S, and if σ' be the density of a superficial distribution on S, which gives rise to the same value, V', of the potential function at each point of S as that of a unit of matter concentrated at any given point O, then the value at O of the potential function due to the first distribution is $\int V' \cdot \sigma \cdot dS$.

136. Show that the derivative of the function $x^2 + xy + z^2$ at the point $(1, 2, 3)$ in the direction defined by the cosines $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\sqrt{2})$ is $\frac{1}{2}(5 + 6\sqrt{2})$. Find the angle between the vector differential parameter of this function and the direction just defined, at any point of the plane $3x + y + 2z\sqrt{2} = 0$, at every point of the line $x + y = 0$, $x = 2z$, and at the origin. Show that it is not possible to find a scalar function the level surfaces of which cut orthogonally the lines of the vector $(x + y, z, y)$. Show that the normal derivative of the function $x^2 + y + z$ with respect to the function $x + y + z$ is zero at every point of the plane $x = -1$. Prove that if u and v are the distances of the point (x, y, z) from two fixed points, $h_u = h_v = 1$.

137. A harmonic function which has a constant gradient different from zero cannot vanish at infinity like the Newtonian Potential Function due to a finite mass.

138. $[f(x, y, z), 0, 0]$, $[\Phi(x), 0, 0]$, $[\Psi(y, z), 0, 0]$, the first of which is neither lamellar nor solenoidal, the second lamellar but not solenoidal, and the third solenoidal but not lamellar, are examples of vectors the lines of which are parallel straight lines, though the intensities are not constant. Prove that if in any region the lines of a vector which is both lamellar and solenoidal are parallel straight lines, the intensity of the vector is everywhere in that region the same.

139. $(2x/r, 2y/r, 2z/r)$ and $(\sin y, \sqrt{3}, \cos y)$, the first of which is lamellar but not solenoidal and the second solenoidal but not lamellar, are examples of vectors with constant intensities, which have lines which are not straight lines parallel to each other. Prove that if the lines of a lamellar point function which has a constant tensor are parallel straight lines, the vector is solenoidal. Prove also that if the lines of a solenoidal vector point function which has a constant tensor are parallel straight lines, the vector is lamellar.

140. The vectors $(x + 2zy, y + 3xz, xy)$, $(2zy, 3xz, xy + 2z)$ have everywhere equal divergences and curls and their tensors

are equal all over the surface $x^2 + y^2 - 4z^2 + 6xyz = 0$. It is evident, therefore, that such vectors as these are not determined when their curls and divergences are given. What additional information would determine an analytic vector which does not vanish at infinity? The scalar potential function of a certain vector has the value unity from $r = 0$ to $r = 1$, where $r^2 \equiv x^2 + y^2 + z^2$; and the value $1/r$ from $r = 1$ to $r = \infty$. Is the vector everywhere solenoidal and lamellar? Can you determine an everywhere lamellar and solenoidal vector which has the value 13 at infinity?

141. If at any surface the normal component or a tangential component of a vector is discontinuous, must we suppose that there is divergence at the surface? Illustrate your answer by a simple numerical illustration.

142. S is a portion of an analytic surface bounded by the closed gauche curve s. S' is a surface which divides space into two portions in each of which the components of a vector Q are represented by analytic functions. At S', some of the components of Q parallel to the surface are discontinuous. S' cuts S in the curve s' which divides S into two portions, S_1 and S_2. Two curves in S_1 and S_2 respectively drawn parallel to s' and very close to it shall be called s_1' and s_2'. K_n shall be the continuous component, in the direction of the normal to S, of the curl of Q. That portion of s which with s_1' embraces practically the whole of S_1 shall be called s_1; that portion of the remainder of s which with s_2' embraces nearly the whole of S_2 is to be denoted by s_2. Apply Stokes's Theorem to S_1 as bounded by s_1 and s_1' and to S_2 as bounded by s_2 and s_2', and show that the line integral of the tangential component of Q around s is not in general accounted for by the surface integral of K_n over S, unless we assign to K_n on s' a value such that its line integral along the line is finite. What is this value? On page 113 Stokes's Theorem is predicated only of analytic vectors. Justify the uses made of the theorem on page 219 and in Sections 82 and 88.

143. Assuming that the surface integral of the normal outward component of any vector taken over any closed surface S, within and on which the vector is analytic, is equal to the volume integral of the divergence of the vector taken throughout the space within the surface, show that if in spherical coördinates R, Θ, Φ are the components of a vector Q, taken in the directions in which r, θ, ϕ increase most rapidly, the divergence of Q is given by the expression

$$D_r(r^2R)\cdot/\,r^2 + D_\theta(\sin\theta\cdot\Theta)\cdot/\,r\sin\theta + D_\phi\Phi\cdot/\,r\sin\theta.$$

144. Assuming that, if ξ, η, ζ are three analytic functions which define a system of orthogonal curvilinear coördinates, and if h_ξ, h_η, h_ζ are the gradients of these functions, the surface integral, taken over any closed surface S, of $U\cdot\cos(\xi, n)$ (where U is any function analytic within and on S, and (ξ, n) is the angle between the exterior normal to S at any point on S, and the direction at that point, in which ξ increases most rapidly) is equal to the volume integral extended through the space enclosed by S, of $h_\xi\cdot h_\eta\cdot h_\zeta\cdot\partial[U/h_\eta\cdot h_\zeta]/\partial\xi$, show that, if Q_ξ, Q_η, Q_ζ are the components in the directions in which ξ, η, and ζ increase most rapidly, of an analytic vector Q, the normal component of Q integrated all over S gives

$$\iiint h_\xi\cdot h_\eta\cdot h_\zeta\left\{D_\xi\left(\frac{Q_\xi}{h_\eta h_\zeta}\right)+D_\eta\left(\frac{Q_\eta}{h_\zeta h_\xi}\right)+D_\zeta\left(\frac{Q_\zeta}{h_\xi h_\eta}\right)\right\}d\tau.$$

Write down an expression for the divergence of an analytic vector in terms of ξ, η, ζ, and, assuming that in the case of spherical coördinates $h_r = 1$, $h_\theta = 1/r$, $h_\phi = 1/r\sin\theta$, show that this yields the result stated in the last problem.

145. Let P_0 be a fixed point and P a movable point in the unlimited region T, without a given surface S, and let P_0P be denoted by r. Show that if a function G' can be found which (1) on S has the value $-1/r$, which (2) is harmonic at P_0 and at every other point of T, and which (3) vanishes at infinity like the Newtonian Potential Function of a finite mass, G' is unique. Show also that if $G \equiv G' + 1/r$, and if w is any function harmonic in T, which vanishes at infinity like a Newtonian Potential Function and has the value w_0 at P_0, $4\pi w_0 = \int w \cdot D_n G\, dS$, where n represents an exterior normal to S. Some writers call G "Green's Function" for the given S and the given P_0; others reserve this name for G'. Attach a physical meaning to G. Define a Green's Function for space inside a closed surface S.

Show that if S is a plane and if r' is the distance of P from the image, in the plane, of the pole P_0, the function G is $1/r - 1/r'$.

146. Show that the expression $\int\int 2\rho_1 \cdot \log(r/r_0) \cdot dA_1$, where r_0 is any constant, might be used for the logarithmic potential function of a columnar distribution of repelling matter.

147. Show that in general the surface density of a charge distributed on a conductor is greatest at points where the convex curvature of the surface of the conductor is greatest.

148. Show that if l, m, n are scalar point functions which define a set of orthogonal curvilinear coördinates in an electric field in air where the potential function is V, and if L, M, N represent the force components taken at every point in the directions in which the coördinates increase most rapidly, $L = -h_l \cdot D_l V$, $M = -h_m \cdot D_m V$, $N = -h_n \cdot D_n V$, and Laplace's Equation can be written

$$D_l(L/h_m \cdot h_n) + D_m(M/h_l \cdot h_n) + D_n(N/h_l \cdot h_m) = 0.$$

149. Prove that if a distribution of electricity over a closed surface produces a force at every point of the surface perpendicular to it, the potential function is constant within the surface.

150. Two conducting spheres of radii 6 and 8 respectively are connected by a long fine wire, and are supposed not to be

exposed to each other's influences. If a charge of 70 units of electricity be given to the composite conductor, show that 30 units will go to charge the smaller sphere and 40 units to the larger sphere, if we neglect the capacity of the wire. Show also that the tension in the case of the smaller sphere is $\frac{25}{288\pi}$ per square unit of surface.

151. The first of three conducting spheres, A, B, and C, of radii 3, 2, and 1 respectively, remote from one another, is charged to potential 9. If A be connected with B for an instant, by means of a fine wire, and if then B be connected with C in the same way, C's charge will be $3 \cdot 6$. [Stone.] If, in the last example, all three conductors be connected at the same time, C's charge will be $4 \cdot 5$.

152. A charge of M units of electricity is communicated to a composite conductor made up of two widely separated ellipsoidal conductors, of semiaxes 2, 3, 4 and 4, 6, 8 respectively, connected by a fine wire. Show that the charges on the two ellipsoids will be $\frac{1}{3}M$ and $\frac{2}{3}M$ respectively. Compare the values of $2\pi\sigma^2$ at corresponding points of the two conductors.

153. Can two electrified bodies attract or repel each other when no lines of force can be drawn from one body to the other?

154. Two conductors, A and B, connected with the earth are exposed to the inductive action of a third charged body. Do A and B act upon each other? If so, how?

155. A spherical conductor A, of radius a, charged with M units of electricity, is surrounded by n conducting spherical shells concentric with it. Each shell is of thickness a, and is separated from its neighbors by empty spaces of thickness a. Show that the limit approached by V_A as n is made larger and larger is $(M/a)\log 2$.

156. The superficial density has the same sign at all points of a conducting surface outside which there is no free electricity.

157. An insulated and uncharged spherical conductor of radius 4 centimetres contains an eccentric spherical cavity the radius of which is 2 centimetres. At the centre of the cavity is a point charge of 10 units. Show that the charges on the inner and outer surfaces are uniformly distributed and that the value of the potential function at all points within the cavity is $10/r - 2.5$.

158. A spherical conductor of 10 centimetres radius is surrounded by a concentric conducting spherical shell of radii 12 centimetres and 15 centimetres. The sphere is at potential zero and the shell at potential unity. Show that the charges are -60, 60, and 15.

159. Prove that the electrical capacity of a conductor is less than that of any other conductor in which it can be geometrically enclosed.

160. Show that two exactly similar conductors symmetrically situated on opposite sides of a plane, so that one is the optical image of the other in the plane, repel each other if raised to the same potential.

161. Prove that the following statements are true: If any conductors, some or all of which are charged, are exposed to one another's influences but are far removed from all other charged bodies, the charge on one, at least, of the conductors must have the same sign throughout. If two charged conductors, uninfluenced except by each other, have equal and opposite charges, the surface density at every point of one has one sign and the surface density at every point of the other the opposite sign. A charge, -1, concentrated at any point P produces a distribution of one sign throughout upon a conductor C which carries a total charge of $1 + \mu$, μ being any positive quantity whatever. If two conductors influenced only by each other are at potentials of the same sign, the distribution has the same sign throughout upon that one of the conductors the potential of which is the greatest in absolute value. If two conductors influenced only by each other

are at opposite potentials, the distribution in each has the same sign everywhere that the potential function has. A charged conductor is always attracted, in the absence of other charged bodies, by every other conductor, in its neighborhood, which is put to earth. [Duhem.]

If n is the number of unit Faraday tubes, per square centimetre, which pass through any small portion of an equipotential surface of an electric field in air, the strength of the field on this small area is $4\pi n$.

162. If when a unit charge is placed on a conductor C in the presence of conductors C_1, C_2, kept at potential zero, the charges on these are $-e_1$, $-e_2$; then if C be discharged and insulated and C_1, C_2 be raised to potentials V_1, V_2, the potential of C will be $e_1V_1 + e_2V_2$.

163. A soap bubble of surface tension T has a charge Q. Show that its diameter is $Q^{\frac{2}{3}}/(2\pi T)^{\frac{1}{3}}$.

164. Prove that the capacity of n equal spherical condensers when arranged in cascade is only about $\frac{1}{n}$th of the capacity of one of the condensers; but that if the inner spheres of all the condensers be connected together by fine wires, and the outer conductors be also connected together, the capacity of the complex condenser thus formed is about n times that of a single one of the condensers.

165. A conductor the equation of the surface of which is

$$\frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{9} = 1$$

is charged with 80 units of electricity; what is the density at a point for which $x = 3$, $y = 3$?

If the density at this point be a, what is the whole charge on the ellipsoid?

166. A charged insulated conductor A is so surrounded by a number of separate conductors B, C, D, $\cdots$, which are put to earth, that no perfectly straight line can be drawn from

any point of A to the walls of the room without encountering one of these other conductors. Will there be any induced charge on the walls of the room?

167. Assuming that in the case of a conductor surrounded by dry air, 800π dynes per square centimetre is the greatest pressure that a charge on the conductor can exert at any point upon the air without breaking down the insulation, show that a spherical conductor must have a diameter of at least 0.126 centimetres in order to hold, in dry air, one electrostatic unit of electricity.

168. Prove that two pith balls each 4 millimetres in diameter and 3 milligrammes in weight, suspended side by side by vertical silk fibres 10 centimetres long, cannot be so highly charged with electricity that the fibres shall make an angle of 60° with each other.

169. Discuss the following passage from Maxwell's *Elementary Treatise on Electricity:*

"Let it be required to determine the equipotential surfaces due to the electrification of the conductor C placed on an insulating stand. Connect the conductor with one electrode of the electroscope, the other being connected with the earth. Electrify the exploring sphere,* and, carrying it by the insulating handle, bring its centre to a given point. Connect the electrodes for an instant, and then move the sphere in such a path that the indication of the electroscope remains zero. This path will lie on an equipotential surface."

170. A condenser consists of a sphere A of radius 100 surrounded by a concentric shell the inner radius of which is 101, and outer radius 150. The shell is put to earth, and the sphere has a charge of 200 units of positive electricity. A sphere B of radius 100 outside the condenser can be connected with the condenser's sphere by means of a fine insulated wire passing

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* A very small conducting sphere fitted with an insulating handle.

through a small hole in the shell. B is connected with A; the connection is then broken, and B is discharged; the connection is then made and broken as before, and B is again discharged. After this process has been gone through with five times, what is A's potential? What would it become if the shell were to be removed without touching A?

$$[2\,(101)^4/(102)^5,\ 2\,(101)^5/(102)^5.]$$

171. If the condenser mentioned in the last problem be insulated and a charge of 100 units of positive electricity be given to the shell, what will be the potential of the sphere? of the shell? If we then connect the sphere with the earth by a fine insulated wire passing through the shell, what will be the charge on the outside of the shell? What will be the potential of the shell? If next A be insulated, and the shell be put to earth, what will be A's potential? What will be its potential if the shell be now wholly removed?

$$[2/3, 2/3, -4040/41;\ 60/41, 2/205, -2/205, -202/205.]$$

172. A conductor is charged by repeated contacts with a plate which after each contact is recharged with a quantity (E) of electricity from an electrophorus. Prove that if e is the charge of the conductor after the first operation, the ultimate charge is $Ee/(E-e)$.

173. If one of a system of n conductors entirely surrounds all the others, $2(n-1)$ of the coefficients of potential have the common value p. If the outside conductor be put to earth, it loses a quantity Q of electricity. Show that the energy loss is $\frac{1}{2}pQ^2$.

174. A conductor is formed of two infinite planes intersecting at right angles and is connected with the earth. A long straight wire, parallel to the intersection of the planes, at distances b and a from them, has an electric charge e per unit length. Show that the electrification of the first

plane at a distance x from the line of intersection of the two is

$$-4\,abex/\pi[(a^2+b^2+x^2)^2-4\,a^2x^2].$$

175. The energy, per unit of surface, of a plane parallel plate condenser in which the superficial charge density is σ_0 is $2\,\pi\sigma_0{}^2a$ when the distance between the plates is a. Show that if the distance be decreased to $a-\Delta a$ the energy is

$$2\,\pi\sigma_0{}^2(a-\Delta a)$$

if the charge remains constant, and

$$2\,\pi\sigma_0{}^2a^2/(a-\Delta a)$$

if the potential remains constant. Hence prove that the rates of change of the energy are equal in value but opposite in sign in the two cases.

176. The foot of the perpendicular dropped from any point P upon the line A_1A_2 shall be marked M. At A_1 is a point charge m_1 and at A_2 a point charge $-m_2$, m_1 being greater than m_2. $A_1P=r_1$, $A_2P=r_2$, $A_1M=x$, $MP=y$, $A_1A_2=a$, $m_1/m_2=/\mu^2$. Show that the surface integral of normal force parallel to the x axis over an infinite plane through M perpendicular to A_1A_2 is $2\,\pi(m_1-m_2)$ if $x>a$; $2\,\pi(m_1+m_2)$ if $0<x<a$; and $2\,\pi(m_2-m_1)$ if $x<0$. The induction outward through an infinite spherical surface with centre at any finite point is $4\,\pi(m_1-m_2)$. Show that the value at any point on a spherical surface of radius r_1, with centre at A_1, of the normal outward component of the force is $m_1/r_1{}^2-m_2\cos(r_1,r_2)/r_2{}^2$, and this is positive for every point of the surface if $r_1>a\mu/(\mu-1)$. It follows from this that no line of force can come from infinity to the charge on A_2; but $4\,\pi m_2$ of the $4\,\pi m_1$ lines which start from A_1 reach A_2. Show that all the lines of force which cross the two planes drawn perpendicular to A_1A_2 through A_1 and A_2 cross them from left to right. The inductions across these planes are $2\,\pi m_2$ and $2\,\pi m_1$. Through M, any point of A_1A_2, imagine a plane drawn perpendicular to

A_1A_2 and let a circumference be drawn on this plane with M as centre and MR as radius. Let the angles which A_1R and A_2R make with the line from A_1 to A_2 be ω_1 and ω_2, then the induction through the circle is $2\pi[m_1(1 - \cos\omega_1) + m_2(\cos\omega_2 - 1)]$ or $2\pi[m_1(1 - \cos\omega_1) + m_2(1 + \cos\omega_2)]$ according as A_1M is greater than a, or positive, and less than a. If in the last case the radius be so chosen that the circle shall include all the lines which converge to A_2, we must equate the induction to $4\pi m_2$. This yields $m_1\cos\omega_1 - m_2\cos\omega_2 = m_1 - m_2$, which may be regarded as the equation of the surface of separation between the lines which go from A_1 to A_2 and those which go to infinity.

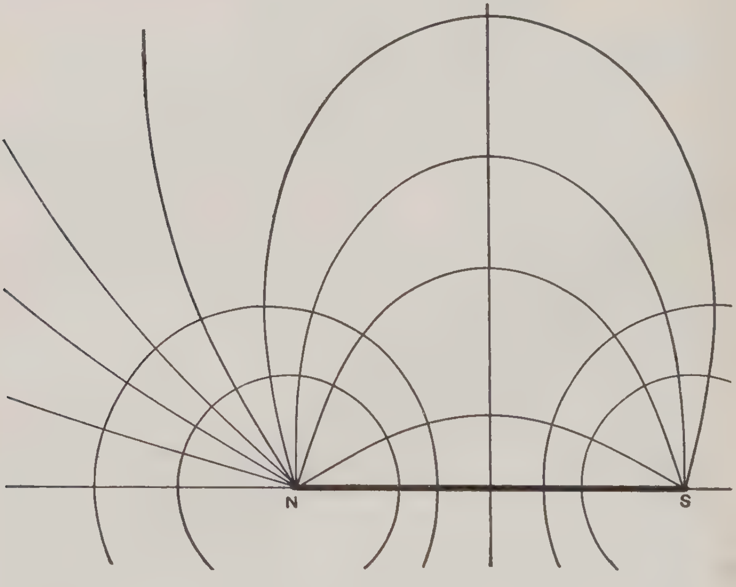

FIG. 125.

$2\pi[m_1(1 - \cos\omega_1) + m_2(1 + \cos\omega_2)] = C$ is the equation of a surface of revolution which includes everywhere C lines of force. Since every meridian curve of this surface is itself a line of force, the equation just written may be regarded as the general equation of the lines of force. If $m_1 = m_2$, the lines are sometimes called "magnetic lines." In this case the equation becomes $\cos \omega_1 - \cos \omega_2 = \text{const.}$, and the lines have the forms of the curves which pass through the points N, S in Fig. 125.

Show that if $\mu = 1$, if R is the resultant force at any point P, and if Q is the point where the line of action of R cuts A_1A_2, $R/[\sin (r_1, r_2)] = m/[r_1^2 \sin (R, r_2)] = m/[r_2^2 \sin (R, r_1)]$ or,

since $$\sin (A_1PQ) = (\sin PQA_1)(A_1Q)/r_1,$$

and $$\sin (A_2PQ) = (\sin PQA_1)(QA_2)/r_2,$$

$$r_1^3/r_2^3 = A_1Q/A_2Q.$$

If Q is fixed, P must move so that r_1/r_2 is constant: its locus is, therefore, a circle. [See Mascart et Joubert, §§ 168 and 169, and also Nipher's *Electricity and Magnetism*, Ch. III.]

177. Two condensers A and B have capacities C_1 and C_2. A is charged by a certain battery and then discharged; it is then charged and its charge is shared with B; finally A and B are both discharged. Show that the energies of the different discharges are to each other as

$$(C_1 + C_2)^2 : C_2(C_1 + C_2) : C_1^2 : C_1C_2.$$

[Clare College.]

178. An earth-connected circular disc 5 centimetres in radius is suspended horizontally from one arm of a balance, and an insulated plate is placed parallel to it and 1 centimetre below it. When the lower plate has been electrified there is found to be an attraction equal to the weight of 1.274 grammes between the two. Show that, assuming the electricity to be uniformly distributed, the potential of the lower plate is about 6000 volts.

179. S is an equipotential surface due to a distribution of matter of which it encloses a portion M_1, and excludes a portion M_2. Let M_1 be distributed on S according to the law $4\pi\sigma = -D_n V$: then superpose on the system thus formed the negative of the original system, so as to have the surface S at zero potential due to the distribution on it and to the negative of M_1 within it. What will now be the value of the potential function without S? At a distance δ_2 from the centre of a spherical cavity of radius r, in a conductor which is at potential zero, is a point charge of m_2 units. Find by aid of the formulas given in Section 65 the density of the charge on the wall of the cavity.

180. If a conductor C, which entirely surrounds a system of charged and insulated conductors, be at first insulated and at potential V, and then put to earth, the potentials of all the interior conductors will be diminished by V. If this system be now discharged, the loss of energy is the same as if C had not been put to earth but had had the interior conductors put into connection with its inner surface. [M. T.]

181. Show that r/δths of the unit Faraday tubes proceeding from an electrified particle, at a distance δ from the centre of a conducting sphere of radius r, which is put to earth, meet the sphere, if there are no other conductors in the neighborhood, and that the rest go off to infinity.

182. If a charge m_1 is placed at a point A_1 distant δ_1 from the centre O of a conducting sphere of radius r (Section 65) kept at potential zero, the charge induced on the surface has the density $\sigma = -m_1(\delta_1^2 - r^2)/4\pi r r_1^3$ at a point distant r_1 from A_1, and the total amount of the induced charge is $-m_1 r/\delta_1$. The attraction between the point charge and the induced charge is $m_1^2 r\,\delta_1/(\delta_1^2 - r^2)^2$. If now a charge M be distributed uniformly over the sphere so as to raise its potential to M/r or V_0, the new density will be

$$\sigma = [V_0 r_1^3 - m_1(\delta_1^2 - r^2)]/4\pi r r_1^3,$$

the new charge $E \equiv V_0 r - m_1 r/\delta_1$, and the attraction

$$F \equiv [m_1^2 r\,\delta_1/(\delta_1^2 - r^2)^2 - m_1 V_0\, r/\delta_1^2],$$

or $$m_1^2 r\,\delta_1/(\delta_1^2 - r^2)^2 - m_1 E/\delta_1^2 - m_1^2\, r/\delta_1^3.$$

This attraction is zero when δ_1 satisfies the equation

$$E\,\delta_1(\delta_1^2 - r^2)^2 = m_1\, r^3(2\,\delta_1^2 - r^2).$$

If $M = +\, m_1 r/\delta_1$, the total charge on the sphere will be zero. In this case $V_0 = m_1/\delta_1$, and the force of attraction is $m_1^2 r^3(2\,\delta_1^2 - r^2)/(\delta_1^2 - r^2)^2\,\delta_1^3$: this expression is always positive. The density on the sphere is zero, if anywhere, on a circumference determined by the equation

$$E + m_1\, r/\delta_1 = m_1\, r(\delta_1^2 - r^2)/r_1^3.$$

O and A_2 are inverse points with respect to a spherical surface S of radius $\sqrt{\delta_1^2 - r^2}$, the centre of which is A_1. If, therefore, T is any point on S, $A_2T \cdot \delta_1 = OT \cdot \sqrt{\delta_1^2 - r^2}$ and, if $M = m_1\, r/\sqrt{\delta_1^2 - r^2}$, the potential function has the same uniform value on S and on the conductor. The intersection of the two surfaces is a line of no force and no density.

The potential function due to m_1 alone is the same as that due to m_1 and the charged sphere, at all points on the spherical surface $OP/A_2P = M\,\delta_1/m_1\, r$: if $E = 0$, this is the plane which bisects A_2O at right angles.

The mutual potential energy of the point charge m_1 and the distribution on the sphere is

$$-\int_{\delta_1}^{\infty} F \cdot d\delta_1 = m_1 E/\delta_1 - \tfrac{1}{2}\, m_1^2 r^3/\delta_1^2(\delta_1^2 - r^2).$$

Show that if a charged conducting sphere of radius 10 centimetres is at potential V_0 in the presence of a point charge of 12 units at a distance of 20 centimetres from the centre of the sphere, the whole charge on the sphere is $10(V_0 - \tfrac{3}{5})$.

Show also that V_0 is $2/15$, $6/5\sqrt{3}$, or 3.6 according as the density of the surface charge is zero, at the point of the surface farthest from A_1, at a point just visible from A_1, or at the point nearest A_1. Show that if the whole charge on the spherical surface is $14/3$ there is no attraction between the point charge and the surface charge; and that if the sphere was originally uncharged and insulated, its potential was constantly equal to $12/\delta_1$ as the point charge gradually approached its present position from infinity.

Show that the integral of $(\delta_1^2 - r^2)/r_1^3$ taken over the surface of the sphere is $4\pi r^2/\delta_1$. How much of the charge on the sphere is visible from A_1?

Find the surface density on a spherical conductor at potential zero under the action of two equal external point charges situated at equal distances on opposite sides of the centre. Consider separately the case where the point charges have opposite signs.

183. An insulated conducting sphere of radius r charged with m units of positive electricity is influenced by m units of positive electricity concentrated at a point $2r$ distant from the centre of the sphere. Give approximately the general shape of the equipotential surfaces in the neighborhood of the sphere.

Give an instance of a positively electrified body the potential of which is negative.

184. Prove that if the spherical surfaces of radii a and b, which form a spherical condenser, are made slightly eccentric, c being the distance between their centres, the change of electrification at any point of either surface is $\frac{3abc \cdot \cos\theta}{4\pi(b-a)(b^3-a^3)}$, where θ is the angular distance of the point from the line of centres, and where the difference between the values of the potential function on the two surfaces is unity.

185. Show that if an insulated conducting sphere of radius a be placed in a region of uniform force (X_0), acting parallel to

the axis of x, the function $-X_0 \cdot x(1 - a^3/r^3) + C$ satisfies all the conditions which the potential function outside the sphere must satisfy, and is therefore itself the potential function. Show that the surface density of the charge on the sphere is $\frac{3\,x\,X}{4\,\pi a}$, and prove that this result might have been obtained by making μ_1 infinite in the formulas near the top of page 206.

186. If q_{11}, q_{22} are the coefficients of capacity of two of a set of conductors, and if q_{12} is their coefficient of mutual induction, the capacity of the compound conductor formed by joining these two conductors by a fine wire is $q_{11} + 2\,q_{12} + q_{22}$, if all the other conductors be put to earth. If p_{11}, p_{22}, p_{12} are the coefficients of potential of the two conductors, and if all the other conductors of the series are uncharged and insulated, the capacity of the compound conductor is

$$(p_{11} + p_{22} - 2\,p_{12}) / (p_{11}p_{22} - p_{21}^2).$$

If the distance b between the centres of two conducting spheres of radii a_1, a_2 is large compared with the diameter of either, $p_{11} = 1/a_1$, $p_{22} = 1/a_2$, and p_{12} is approximately $1/b$, so that if e_1, e_2 are the charges of the spheres and V_1, V_2 their potentials, $V_1 = e_1/a_1 + e_2/b$, $V_2 = e_1/b + e_2/a_2$. Show that, approximately,

$$q_{11} = a_1b^2/(b^2 - a_1a_2), \quad q_{12} = -\,a_1a_2b/(b^2 - a_1a_2),$$
$$q_{22} = a_2b^2/(b^2 - a_1a_2).$$

187. If on the radius vector OP drawn from a fixed point O to another point P a new point P' be taken, such that $\overline{OP} \cdot \overline{OP'} = a^2$, where a is a constant chosen at pleasure, P and P' are said to be *inverse points*, O is the *centre of inversion*, a sphere of radius a with centre at O is the *sphere of inversion* and a the *radius of inversion.* One of a pair of inverse points is without the sphere of inversion and the other within, unless both coincide. The straight line which

joins the points of contact of tangents to the sphere drawn from an outside point P' passes through the inverse point P. If P, P' and Q, Q' are pairs of inverse points, the triangles OPQ and $OQ'P'$ are similar. If one (P) of a pair of inverse points moves along a curve, or over a surface, or through a space, the other (P') will generate the inverse curve, surface, or space. A plane at a perpendicular distance b from O inverts into a spherical surface of radius $a^2/2\,b$, passing through O. A spherical surface of radius c with centre at a distance b from O inverts into another spherical surface of radius $a^2c/(b^2-c^2)$ with centre at a distance $a^2b/(b^2-c^2)$ from O. If $a^2=b^2-c^2$, the spherical surface inverts into itself, though the inverse of the old centre is not the new centre. The centre of inversion inverts into the region at infinity.

Prove that if the origin be the centre of inversion, a point P or (x, y, z), distant r from the origin, inverts into a point P' or (x', y', z'), distant r' from the origin, where $rr'=a^2$, $x/r=x'/r'$, $y/r=y'/r'$, $z/r=z'/r'$, $x=a^2x'/r'^2$, $y=a^2y'/r'^2$, $z=a^2z'/r'^2$, $x'=a^2x/r^2$, $y'=a^2y/r^2$, $z'=a^2z/r^2$. An element of arc ds at P inverts into an element of arc ds' at P', such that $ds=r^2\cdot ds'/a^2=a^2\cdot ds'/r'^2$. An element of area dS at P inverts into an element of area dS' at P', such that $dS=r^4\cdot dS'/a^4=a^4\cdot dS'/r'^4$. An element of volume $d\tau$ at P inverts into an element of volume $d\tau'$ at P', such that $d\tau=r^6\cdot d\tau'/a^6=a^6\cdot d\tau'/r'^6$. The angle between two curves which intersect at P is equal to the angle between the inverse curves which intersect at P'. If P and P' be drawn in different diagrams, in which the rectangular Cartesian coördinates are x, y, z and x', y', z' respectively, $\xi\equiv a^2x'/r'^2$, $\eta\equiv a^2y'/r'^2$, $\zeta\equiv a^2z'/r'^2$, define a set of orthogonal curvilinear coördinates in the second diagram, and the Cartesian coördinates of P in the first diagram are equal to the curvilinear coördinates of P' in the second diagram. Any function $F(x, y, z)$ has the same numerical value at P that the function $F(a^2x'/r'^2,\ a^2y'/r'^2,\ a^2z'/r'^2) \equiv \psi(x', y', z')$ has at P'. Prove that

$$(D_x^2 + D_y^2 + D_z^2)\, F(x, y, z)_{\text{at } P}$$
$$= (r'^5/a^5)\,(D_{x'}^2 + D_{y'}^2 + D_{z'}^2)\,(a\psi/r')_{\text{at } P'.}$$

If F is zero on any surface or throughout any space in the first diagram, $a\psi/r'$ is zero on the corresponding surface or throughout the corresponding space. If F has the constant value c on the surface S, $a\psi/r'$ has the value ac/r', which is not constant on the corresponding surface S'.

If F is the potential function due to a volume distribution of density ρ in a region T, together with a superficial distribution of density σ on a surface S and a point charge e at a point Q, $(a\psi/r')$ is the potential function due to a volume distribution of density $\rho' = a^5\rho/r'^5$ in the region T', corresponding to T, together with a superficial distribution of density $\sigma' = a^3\sigma/r'^3$ on the surface S', which corresponds to S, and a point charge $e' = r'e/a$ at the point Q', which is the inverse of Q. The inverse of a point charge e at the centre of inversion is a charge at infinity, which raises all finite points to potential e/a. If F is the potential function of a distribution σ, ρ which keeps a certain surface S at potential zero, $(a\psi/r')$ will be the potential function of a distribution σ', ρ' which keeps the corresponding surface S' at potential zero. If F is the potential function of a distribution σ, ρ which keeps the surface S at potential c, $(a\psi/r')$ will be the potential function of a distribution σ', ρ' which keeps S' at the potential ac/r': if, however, we add to the distribution σ', ρ' a point charge $-ac$ at the origin, the new potential function will keep S' at potential zero.

188. Show that if a point charge e be anywhere between two infinite planes which form a diedral angle of 60°, these planes would form a surface of potential zero due to the original charge and five images in the planes. Find the density of the charge on two planes which form an angle π/n,

if they are kept at potential zero in presence of a point charge between them. Invert the system with respect to the charged point.

189. A homogeneous sphere of density ρ and radius c has its centre at a point C distant d from an outside point O. The value of the potential function at a point P outside the sphere is $\frac{4}{3}\pi\rho c^3/CP$. Show that if the distribution be inverted, using O as centre, the new distribution is a heterogeneous, centrobaric sphere of mass $\frac{4}{3}\pi\rho c^3 a/d$, the baric centre of which is the inverse point of C. [Routh.]

190. A point charge $+e$ lies on the x axis at a distance $+b$ from the origin between two conducting plates, $x = 0$, $x = 2c$, both of which are kept at zero. Show that the images of the point charge in the planes are an infinite series of point charges numerically equal to e but alternately positive and negative at points on the x axis. The coördinates of the negative images are $-b, -(4c+b), -(8c+b), \cdots, (4c-b), (8c-b), (12c-b), \ldots$, and those of the positive images are $(4c+b), (8c+b), (12c+b), \cdots, -(4c-b), -(8c-b), -(12c-b), \cdots$. Show that the force at any point between the planes might be computed from these images and the original point charge. Indicate a method for determining the density of the induced charges on the plates. State clearly the result of inverting the system, using the original charged point as centre of inversion, and each of several different values for a.

If in this problem the charge e is at a point O midway between the plates, and if this point be chosen as origin, $b = c$, and there are positive images at points the x coördinates of which are $0, 4c, 8c, 12c, \cdots, -4c, -8c, -12c, \cdots$, and negative images at points the x coördinates of which are $-2c, -6c, -10c, \cdots, 2c, 6c, 10c, \cdots$. Show that if the system be inverted, using O as centre of inversion and c as radius of inversion, and if the inverse of the charge at O be omitted, the result is a conductor formed of two spherical surfaces of radius $r = \frac{1}{2}c$, in contact at potential $V_0 \equiv -e/c$ under positive charges equal respectively to $\frac{1}{4}e$, $\frac{1}{8}e$, $\frac{1}{12}e, \cdots$, at points the x coördinates of which are $\pm\frac{1}{4}c$, $\pm\frac{1}{8}c$, $\pm\frac{1}{12}c, \cdots$; and negative charges $\frac{1}{2}e$, $\frac{1}{6}e$, $\frac{1}{10}e, \cdots$, at points the x coördinates of which are $\pm\frac{1}{2}c$, $\pm\frac{1}{6}c$, $\pm\frac{1}{10}c, \cdots$. The total charge in each of the spheres is

$$-\tfrac{1}{2}e\,(1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}\cdots) = -\tfrac{1}{2}e\cdot\log 2 = V_0 r\cdot\log 2,$$

and their mutual repulsion, $\frac{1}{6}V_0^2(\log 2 - \frac{1}{4})$.

191. If two spherical conductors each of radius a have charges e_1, e_2, and are at a great distance apart, the energy of the system is $(e_1^2 + e_2^2)/2a$. If the two are brought up into contact, the whole charge of the compound conductor thus formed is $(e_1 + e_2)$, it is at potential $(e_1 + e_2)/2a\cdot\log 2$, and the energy of the system is $(e_1 + e_2)^2/4a\cdot\log 2$. Show that the work done against the mutual repulsions of the two charges during the approach of the spheres is about

$$[(0.722)\,e_1e_2 - (0.139)\,(e_1^2 + e_2^2)]/a,$$

and discuss separately the special cases

$$e_1 = 0,\ e_1 = e_2,\ e_1 = 5\,e_2,\ e_1 = \tfrac{3}{2}\,e_2.$$

192. Show that if a point charge be situated at a point O, between two concentric spherical surfaces, it is possible to find a series of electric images which together with the original charge would keep each of the surfaces at potential zero. What would be the result of inverting the system, using O as centre?

193. A certain condenser consists of a closed conducting surface S_1 surrounded by another closed conducting surface S_2, separated from the first by a homogeneous dielectric. When the condenser is charged, the lines of force between S_1 and S_2 are the same as if S_2 were removed and S_1 freely charged. What do you know about S_1 and S_2?

194. The semiaxes of a conducting prolate ellipsoid of revolution are 10, 8, 8. Find, by help of the formulas of Sections 6 and 23, the external field when the conductor has a

free charge of 60 units, and show that the surface density at the equator is then $3/16\pi$.

195. A prolate conducting spheroid of major axis $2a$ and minor axis $2b$, has a charge of electricity E. Prove that the attraction between the two halves into which it is divided by its diametral plane is $E^2 \cdot \log(a/b)/4(a^2 - b^2)$. [St. John's College.]

196. If a particle charged with a quantity e of electricity be placed at the middle point of the line joining the centres of two equal spherical conductors kept at zero potential, the charge induced on each sphere is

$$-2em(1 - m + 2m^2 - 3m^3 + 4m^4 \cdots),$$

where m is the ratio of the radius of either of the spheres to the distance between their centres.

197. A conducting sphere of small radius a is situated in the open air at a considerable height h above the ground. Show that its electrical capacity is increased by the neighborhood of the ground in the ratio of $1 + \left(\frac{a}{2h}\right)^2$ to 1, very nearly.

198. A negative point charge, $-e_2$, lies between two positive point charges e_1 and e_3 on the line joining them and at distances a and b from them respectively. Show that if

$$\frac{e_1}{b} = \frac{e_3}{a} = \frac{e_2\lambda^3}{a+b}, \quad \text{where} \quad 1 < \lambda^2 < \left(\frac{a+b}{a-b}\right)^2,$$

there is a circumference at every point of which the force vanishes.

199. Two spherical conducting surfaces of radii a and b form a condenser. Prove that if the centres be separated by a small distance d, the capacity is approximately

$$\frac{ab}{b-a}\left\{1 - \frac{abd^2}{(b-a)(b^3-a^3)}\right\}.$$

When $d = 0$, the capacity is $\frac{ab}{b-a}$.

200. A small insulated conductor, originally uncharged, is connected alternately with two insulated conductors A and B at a considerable distance apart. Prove that if e_0 and e_0' are the original charges of A and B, e_1 and e_1' their charges after the carrier has touched A and then touched B, the charge of B when the carrier has touched A and B each n times is

$$\frac{ab - e_1' - e_0'}{ab - 1} + \frac{e_0' - e_1'}{(ab-1)\,a^{n-1}b^{n-1}},$$

where $a = e_0/e_1$ and $b = (e_0 + e_0' - e_1)/e_1'$.

The charges of A, B, and the carrier, are ultimately in the ratios

$$e_1(e_0 - e_1 + e_0' - e_1') : e_1'(e_0 - e_1) : (e_0 - e_1)(e_0 - e_1 + e_0' - e_1').$$

201. If a series of conductors were constructed which might be made to coincide with the closed level surface of a harmonic function w which vanishes at infinity like a Newtonian Potential Function, the capacities of any two of these conductors would be to each other in the ratio of the reciprocals of the values of w on the corresponding surfaces. If two of the surfaces for which $w = w_1$ and $w = w_2 < w_1$ be constructed of metal, and if charges E_1 and E_2 be given them, the energy is

$$\frac{1}{2}\left\{\frac{E_1^2(C_2 - C_1)}{C_1C_2} + \frac{(E_1 + E_2)^2}{C_2}\right\},$$

where C_1 and C_2 are the capacities. The energy becomes $\frac{1}{2}(E_1 + E_2)^2/C_2$ if the two are connected.

202. An insulated conducting sphere of radius r, bearing a charge m, is introduced into a field of force due to a fixed distribution M of electricity. Show that if the value of the potential function due to M at the centre of the sphere is C, the value of the potential function within the sphere is $C + m/r$.

203. Compute the force at the point (x, y, z) due to a particle of mass $-m$ at the point $(a, 0, 0)$ and a particle of mass

$+m$ at the point $(-a, 0, 0)$, and show that if m and a be made to increase indefinitely in such a manner that the ratio of m to a^2 is always equal to the constant A, the field becomes ultimately a uniform field of intensity $X = 2A$.

204. It is evident that the value at any point P in the xy plane, of the potential function due to two slender, infinitely long, homogeneous, straight filaments (of mass m and $-m$ respectively per unit length) which cut the plane perpendicularly at the points A and B, is $2m \log (AP/BP)$, and that the equipotential surfaces are circular cylinders [one is a plane] such that A and B are inverse points with respect to every one of them. If the radius of any one of these cylindrical surfaces the axis of which cuts the xy plane at C be denoted by r (Fig. 126), and if $AP = r_1$, $BP = r_2$, $AC = \delta_1$, $BC = \delta_2$, $\delta_1/r = r/\delta_2 = r_1/r_2$, and the triangles BCP and ACP are similar if P lies on the cylinder. The resultant force F at P has the direction of the normal to the cylinder, the repulsion due to the filament which cuts the plane at A is $2m/r_1$, and the attraction due to the filament which cuts the plane at B is $2m/r_2$. If the angle APB be denoted by α, the Principle of the Parallelogram of forces applied at P yields

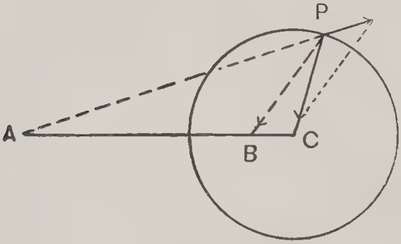

FIG. 126.

$$F/\sin\alpha = 2m/[r_2 \sin(r, r_1)] = 2m/[r_1 \sin(r, r_2)],$$

and the Theorem of Sines applied to the triangle APB yields

$$AB/\sin\alpha = r_1/\sin(r, r_1) = r_2/\sin(r, r_2),$$

so that

$$F = 2mAB/r_1 \cdot r_2 = 2m\delta_1 AB/r \cdot r_1^2 = 2m\delta_2 AB/r \cdot r_2^2$$
$$= 2m(\delta_1^2 - r^2)/r \cdot r_1^2 = 2m(r^2 - \delta_2^2)/r \cdot r_2^2.$$

The value of the potential function on the cylinder is

$$2\,m \log \delta_1 / r = 2\,m \log r / \delta_2.$$

If, now, the mass of the filament which cuts the plane at B be spread on the cylindrical surface so that the surface density at every point is

$$\sigma = -F/4\pi \text{ or } -m(\delta_1^2 - r^2)/2\pi r \cdot r_1^2,$$

the potential function outside the cylinder will be unchanged. If, finally, a mass m' per unit length be spread uniformly over the cylinder, the value of the potential function within and on the surface will be $2\,m \log(\delta_1/r) + 2\,m' \log r$, and, by a suitable choice of m', this may be given any value. The whole charge on the unit length of the cylindrical surface is $m' - m = M$, the value of the potential function on the surface is $V_S = 2\,m \log(\delta_1/r) + 2(M + m) \log r$, and the surface density at a point distant r_1 from the straight line which cuts the paper perpendicularly at A is

$$(M + m)/2\pi r - m(\delta_1^2 - r^2)/2\pi r \cdot r_1^2.$$

At any point Q without the cylinder the value of the potential function is

$$2\,m \log(AQ/BQ) + 2(M + m) \log CQ.$$

Show that the force of attraction between the charge on the cylinder and the unit length of the filament through A is

$$2\,m\,[m\delta_1/(\delta_1^2 - r^2) - (M + m)/\delta_1].$$

This force vanishes if $\delta_1^2/r^2 = (M + m)/M$. Show also that if the cylinder is at potential zero in the presence of the filament through A, $M = -m \log(\delta_1/r)$.

If we superpose upon this distribution a second consisting of a homogeneous filament of mass $-m$ per unit of length and cutting the paper perpendicularly at A and a similar filament of mass $+m$ per unit length cutting the paper at B, and notice that the potential function due to the new distribution has the value $-2m\log(\delta_1/r)$ at every point of the cylindrical surface, we shall see that the potential function due to the two distributions has at any point Q outside the cylinder the value $2m'\log(CQ) = 2(M+m)\log CQ$ and at any point within the value

$$2m'\log r + 2m\log\delta_1/r - 2m\log(r_1/r_2)$$
$$= 2(M+m)\log r + 2m\log\delta_1 r_2/rr_1.$$

On the cylindrical surface the potential function has the constant value $2(M+m)\log r$, and the surface density at any part of it is, as before, $(M+m)/2\pi r - m(\delta_1^2 - r^2)/2\pi rr_1^2$, or $(M+m)/2\pi r + m(\delta_2^2 - r^2)/2\pi rr_2^2$. What is the physical meaning of the special case where $M + m = 0$?

205. Let $\phi(x, y)$ be a logarithmic potential function due to a body distribution of density ρ through an infinite cylinder the right section of which made by the xy plane is the region T, together with a superficial distribution of density σ on an infinite cylindrical surface the right section of which is the curve s in the xy plane. Let $\alpha = f_1(x, y)$ and $\beta = f_2(x, y)$ be any two conjugate functions analytic in the region considered, and form arbitrarily the new function $\Phi(x, y) \equiv \phi[f_1(x, y), f_2(x, y)]$ by substituting for x and y in ϕ, α and β respectively. To avoid confusion call the rectangular Cartesian coördinates in the plane in which T and s are drawn α and β, instead of x and y, and draw a new xy plane in which to study the new function Φ. In this second figure the curves in which $\alpha = f_1(x, y)$, $\beta = f_2(x, y)$ are constant form a set of orthogonal curvilinear coördinates. A point P which in the first diagram has Cartesian coördinates (α_0, β_0) is said to be transformed into a point P' in the new diagram, the curvilinear coördinates of which are (α_0, β_0). The Cartesian coördinates of P' are (x_0, y_0), where $f_1(x_0, y_0) = \alpha_0$, $f_2(x_0, y_0) = \beta_0$. It is evident that $\Phi(x, y)$ has the same

numerical value at P' that $\phi(\alpha, \beta)$ has at P. The points which lie on the curve s in the old diagram are transformed into points which lie on a curve s' in the new diagram, so that the curve s is transformed into the curve s' and, similarly, the region T into the region T'. It is evident from the properties of conjugate functions that two curves which cut at an angle θ at a point P in the old diagram transform into two curves which cut each other, in general, at the same angle at the point P'. Show that Φ is the logarithmic potential function due to a body distribution through the infinite cylinder of which T' is the cross-section, together with a surface distribution on the cylindrical surface of which s' is the trace. Show also that if h^2 represents either of the two equal quantities $(D_x\alpha)^2 + (D_y\alpha)^2$, $(D_x\beta)^2 + (D_y\beta)^2$, the numerical relations, at corresponding points in the two diagrams, of the corresponding elements of arc and area, of the corresponding values of the volume and surface density, etc., etc., are truly given by the equations

$$ds = h\,ds';\ dA = h^2\,dA';\ \rho h^2 = \rho';\ \sigma h = \sigma';$$

$$\phi = \Phi;\ \Delta\phi = \Delta\Phi;\ h\,D_s\phi = D_{s'}\Phi,\ h^2\Delta^2\phi = \Delta^2\Phi;$$

$$h\,D_n\phi = D_{n'}\Phi;\ \rho\,dA = \rho' dA';\ \sigma\,ds = \sigma' ds'.$$

206. Given in a plane two circles of radii a and b respectively, which have no points in common, it is possible to find two points (Q_1, Q_2) on the line which joins their centres (A, B), such that if r_1 and r_2 represent the distances from Q_1 and Q_2 of any moving point, both circles belong to the family of curves represented by the equation $r_1/r_2 = c$. Show that if $AB \equiv d$, and if the circles are mutually exclusive, Q_1 and Q_2 are between A and B, and

$$AQ_1 = (a^2 + d^2 - b^2 - R)/2\,d,\ \ BQ_2 = (b^2 + d^2 - a^2 - R)/2\,d,$$

where $R^2 = (a^2 - b^2 - d^2)^2 - 4\,b^2d^2$. If one of the circles lies within the other and if $a > b$, Q_1 and Q_2 lie beyond B on the line AB, Q_1 is within both circles, and Q_2 outside both. In this case,

$$AQ_1 = (a^2 - b^2 + d^2 - R')/2\,d, \quad BQ_2 = (a^2 - b^2 - d^2 + R')/2\,d,$$

where $R'^2 = (b^2 - a^2 - d^2)^2 - 4\,a^2d^2$. In the second case the four points of contact of tangents drawn from Q_2 to the two circles lie on a straight line through Q_1. What is the corresponding fact in the case first treated? Consider the special case where $a = b = \frac{1}{2}\,d$.

Prove that the values of AQ_1, BQ_2 given in the subjoined table are correct and draw to scale a diagram for each of the four examples.

a	b	d	AQ_1	BQ_2
1	2	4	0.35	1.09
4	2	1	1.37	10.62
3	1	1	1.14	6.85
5	3	1	1.62	14.38

Given a circle of radius a, with centre at A, and a straight line in its plane at a distance d from A greater than a; the line and the circumference belong to the family of curves $r_1/r_2 = c$, where r_1 and r_2 represent the distances from two points (Q_1, Q_2) equidistant from the line on opposite sides of it and lying on the perpendicular to the line drawn through A. Show that if $Q_1Q_2 = 2\,m$, $m^2 = d^2 - a^2$.

207. If $r_1^2 \equiv (x+a)^2 + y^2, r_2^2 \equiv (x-a)^2 + y^2, \tan\theta_1 \equiv y/(x+a)$, $\tan\theta_2 \equiv y/(x-a)$; $\phi \equiv A\log(r_1/r_2)$, and $\psi \equiv A(\theta_1 - \theta_2)$ are conjugate functions, and if, moreover,

$$c^2 \equiv e^{2\phi/A}, \; \alpha \equiv \frac{a(c^2+1)}{c^2-1}; \quad \alpha - a = \frac{2\,a}{c^2-1}, \text{ and } r \equiv \pm\frac{2\,ac}{c^2-1},$$

where the upper or lower sign is to be used according as c^2 is greater or less than unity; $(x - \alpha)^2 + (y - 0)^2 = r^2$, and the curves of constant ϕ are the circles surrounding the points $(a, 0)$ and $(-a, 0)$ represented in Fig. 59. Values of c

between 0 and 1 correspond to circles which lie to the left of the y axis, and positive values of c greater than 1 to circles on the right of the y axis.

When $\quad c > 1,\ c = (\alpha + a)/r$, and $a = +\sqrt{\alpha^2 - r^2}$,

but when $\quad c < 1,\ c = -(\alpha + a)/r,\ a = +\sqrt{\alpha^2 - r^2}$.

On two circumferences of the system, of equal radii, on opposite sides of the y axis, r_1/r_2 has reciprocal values.

Using these formulas, prove that the charge per unit length on a long cylindrical wire of radius 0.5 centimetre, kept at potential unity at an axial distance of $\alpha = 600$ centimetres from an infinite plane kept at potential zero, is 0.06424 units. In this case c is about 2400, a about 599.9998, and A, 0.128. Show also that if $r = 0.5$ and $\alpha = 10$; $a = 9.988$, $c = 39.975$, $A = 0.271$, but that if $r = 0.5$ and $\alpha = 1$; $a = 0.866$, $c = 3.732$, $A = 0.759$. It is to be noticed that 300 volts are equivalent to 1 electrostatic unit of potential difference, 1 microfarad to 900,000 electrostatic units of capacity, 1 ohm to $1/(9 \times 10^{11})$ electrostatic units of resistance, 1 ampere and 1 coulomb to 3×10^9 corresponding electrostatic units.

In general, if an infinite conducting cylinder of revolution kept at potential V_0 be placed with its axis parallel to an infinite conducting plane at a distance α from it, the charge per unit of length is $\frac{1}{2} V_0 / \log \dfrac{\alpha + \sqrt{\alpha^2 - r^2}}{r}$, and the surface density is inversely proportional to the distance from the plane.

208. A condenser is formed by two long conducting circular cylinders, one of which is entirely inside the other. Prove that if r and r' are the radii, d the distance between the axes, and $2a$ the distance between the limiting points of the coaxial system to which the cylinders belong, the inverse of the capacity per unit length is

$$2 \log \frac{r'(r'^2 - r^2 - d^2 + 2ad)}{r(r'^2 - r^2 + d^2 + 2ad)}. \qquad \text{[St. John's College.]}$$

209. Electricity is distributed in equilibrium over the surface of an infinitely long right cylinder, the cross-section of which is $x^4 + y^4 = a^4$. Prove that the attraction at any external point (r, θ) is inversely proportional to

$$\left(r^4 + 2\,a^4 \cos 4\,\theta + \frac{a^8}{r^4}\right)^{-\frac{1}{4}}$$

and that its direction makes with the axis of x an angle $\frac{1}{2} \tan^{-1}\left(\frac{r^4 - a^4}{r^4 + a^4} \tan 2\,\theta\right)^{\frac{1}{2}}$. [Clare College.]

210. A long, right circular cylinder of radius a is placed with axis parallel to a plane at potential zero. Show that the mutual attraction per unit length of the cylinder between it and the plane is $E^2/\sqrt{c^2 - a^2}$, where c is the distance of the axis of the cylinder from the plane and E the quantity of electricity on the unit length of the cylinder. [M. T.]

211. A_1, Q, A_2, three points in order on a straight line, such that $A_1Q = m$, $QA_2 = n$, have charges

$$e_1 \equiv \lambda \sqrt{m\,(m + n)},\; e_0 \equiv - \lambda \sqrt{mn},\; e_2 \equiv \lambda \sqrt{n\,(m + n)}$$

respectively. The charges e_0, e_2 produce potential zero on a spherical surface S_1 of radius $a \equiv \sqrt{m\,(m + n)}$ with centre at A_1, and the charges e_1, e_0 produce potential zero on a spherical surface S_2 of radius $b \equiv \sqrt{n\,(m + n)}$ with centre at A_2. S_1 and S_2 cut each other orthogonally and together form the equipotential surface λ due to e_0, e_1, and e_2. Show, by a method analogous to that of Section 65, that the resultant force at any point P of S_1, due to e_0 and e_2, is directed towards A_1 and is numerically equal to $\lambda b^3 / a \cdot \overline{A_2P}^3$, so that the whole force at P has the direction A_1P and the intensity

$$F_1 \equiv \lambda [1 / a - b^3 / a \cdot \overline{A_2P}^3].$$

Find a similar expression for the whole force at any point of the surface S_2. Show by the help of Section 31 that the surface integrals, taken over the larger segment of S_1, of the normal components of the forces due to e_1, e_0, and e_2 respectively are $2\pi e_1(1+m/a)$, $2\pi e_0$, $2\pi e_2(1-n/b)$. Prove that if e_0, e_1, and e_2 were distributed on the surface composed of the larger segments of S_1 and S_2 according to the law $\sigma = F/4\pi$, the surface would be at potential λ, and there would be no density at the circle of intersection of S_1 and S_2. The charge under these circumstances on the larger segment of S_1 would be

$$\tfrac{1}{2}\left[e_1(1+m/a)+e_0+e_2(1-n/b)\right],$$

or

$$\tfrac{1}{2}\lambda(a+b+m-\sqrt{mn}-n),$$

or

$$\tfrac{1}{2}\lambda b\left[1+\delta+(\delta^2-\delta-1)/\sqrt{1+\delta^2}\right],$$

where $\delta \equiv a/b$. If b is very large compared with a, the larger segment of S_1 becomes nearly hemispherical; its charge is about $3\lambda a^2/4b$ and its mean density $3\lambda/8\pi b$. The mean density on S_2 when the ratio of a to b is small is nearly equal to $\lambda(4b^2-3a^2)/16\pi b^3$. If $a/b=0$, we have a hemispherical boss on an infinite plane; the ratio of the average densities of the charges on the boss and the plane is $3/2$.

212. A point charge e at $(4b, 0, 0)$ and a point charge $-e$ at $(-4b, 0, 0)$ keep the plane $x=0$ at potential zero. Show that if the system be inverted, using the point $(-2b, 0, 0)$ as centre of inversion and $2b$ for radius of inversion, we obtain a spherical surface of radius b, with centre at $(-b, 0, 0)$, kept at potential zero by the charge $-e$ at the point $(-4b, 0, 0)$, and the charge $\frac{1}{3}e$ at $(-\frac{4}{3}b, 0, 0)$: this is the problem of Section 65. If the centre of inversion were $(-4b, 0, 0)$ and if a were $4b$, we should obtain by inversion a spherical surface of radius $2b$, with centre at $(-2b, 0, 0)$ at potential zero, under a charge $\frac{1}{2}e$ at its centre, and an infinite charge at infinity which lowers the potential function at all finite points by $e/4b$. If this last were omitted, the value of the potential function on the spherical surface would be $e/4b$, as is otherwise evident. Invert a spherical surface uniformly charged with density σ, using any point not its centre as centre of inversion.

213. If $z = w + e^w$; $x = \phi + e^\phi \cos\psi$, $y = \psi + e^\phi \sin\psi$, the slopes of curves of constant ϕ are given by the equation

$$dy/dx = -(1 + e^\phi \cos\psi)/e^\phi \sin\psi,$$

and the slopes of curves of constant ψ by the equation

$$dy/dx = e^\phi \sin\psi/(1 + e^\phi \cos\psi).$$

Show that: (1) The curve $\psi = 0$ is the axis of x, and on it $x = \phi + e^\phi$; large positive values of x and ϕ correspond, and large negative values correspond. (2) If $\psi = \pi$, $y = \pi$ and $x = \phi - e^\phi$: the maximum value of x is -1, so that the curve $\psi = \pi$ is so much of the line $y = \pi$ as lies to the left of $x = -1$. (3) The curve $\psi = -\pi$ is so much of the line $y = -\pi$ as lies to the left of $x = -1$. (4) The curve $\phi = 0$ has the slope $-\text{ctn}\,\frac{1}{2}\psi$ and passes through the points $(-1, \pi)$, $(-1, -\pi)$, $(1, 0)$, $(0, \frac{1}{2}\pi + 1)$, $(\frac{1}{2}, \frac{1}{3}\pi + \frac{1}{2}\sqrt{3})$. (5) The curve $\phi = -m$, where m is any positive quantity, lies to the left of $\phi = 0$, between the lines $y = -\pi$, $y = +\pi$: it has the slope $-(e^m + \cos\psi)/\sin\psi$, which is infinite when $y = 0$ or $y = -\pi$ or $y = \pi$, and has the minimum value $\sqrt{e^{2m} - 1}$, so that for values of m greater than 3 the line is hardly distinguishable from a straight line parallel to the axis of y, at a distance of $(e^{-m} - m)$ to the left. (6) The curve $\phi = +m$, where m is a positive quantity, lies to the right of the curve $\phi = 0$, it cuts the axis of y perpendicularly and meets (but does not cross) the lines $y = -\pi$, $y = \pi$ from without. The curves for which ϕ has the values 1, 2, 3, 4, 5, 6, 7 meet $y = \pi$ at points the abscissas of which are -1.72, -5.39, -17.08, -50.60, -143.4, -397.5, -1090 respectively. (7) If so much of the planes $y = +\pi$, $y = -\pi$ as lie to the left of $x = -1$ be considered conducting and be charged to potential $+\pi$ and $-\pi$ respectively, ψ represents the potential function in the air near them. In this case the charge on either side of a strip of either plane between the planes $x = -x_1$, $x = -x_2$ is equal, per unit length of the strip parallel to the z axis, to the difference on that side of the plane between the values of ϕ for $x = -x_1$ and $x = -x_2$, divided by 4π. On a strip of the plane $y = \pi$, between $x = -1$ and $x = -50.6$, there is, per unit height of the strip, a charge $1/\pi$ on the upper side and of $50.6/4\pi$ on the under side: the charge on corresponding portions of $y = -\pi$ being equal and opposite to these. [Helmholtz, *Crelle's Journal*, Vol. LXX.]

State carefully some problem in electrostatics which might be solved by the use of the function $z = A[cw + e^{cw}]$.

A condenser consists of two very thin, large, plane, metal sheets of the same area parallel to each other at a distance of 1 millimetre. The dielectric is air and the difference of potential between the plates is 1 electrostatic unit (300 volts). Show that the density of the charge 2 millimetres from the edge is about $5/2\pi$ per square centimetre on the inside of the plate.

Discuss at length the function

$$z \equiv \frac{n^n(1-n)^{1-n}}{\sin(n\pi)}\left(e^{(1-n)w} - e^{-nw}\right),$$

where n is any real constant between 0 and $\frac{1}{2}$ [Harris, *Ann. of Math.*, 1901], and state some problems of electrostatics which can be solved by its aid.

214. Three closed surfaces 1, 2, 3 in order are equipotential surfaces of an electrostatic field in air. If an air condenser were constructed with the faces 1, 2, its capacity would be A, but if the faces were 2, 3, its capacity would be B. Show that if a condenser were constructed with faces 1, 3 while a homogeneous dielectric of inductivity μ filled the space 1, 2, and a second dielectric of inductivity μ' the space 2, 3, the capacity of this condenser would be C, where $1/C = 1/\mu A + 1/\mu' B$.

215. A condenser is formed of two concentric spherical conducting surfaces of radii a and c, separated by two dielectric shells bounded by a spherical surface of radius b concentric with the conducting surfaces. Prove that if in one shell

$\mu = m/r^2$ and in the other m'/r^2, the capacity of the condenser is $mm'/[m'(b-a) + m(c-b)]$.

216. If the space between two closed equipotential surfaces in air be filled with a dielectric the inductivity of which is either uniform or a scalar point function the level surfaces of which coincide with the equipotential surfaces of the field, the potential function without the shell will be unchanged, but its value within will be increased by a constant.

217. An infinite dielectric is bounded by an infinite conducting plane which is maintained at a potential λr^2, where r is the distance from a point O in the plane. Prove that if the inductivity of the dielectric varies as the distance z from the plane, the potential at any point is $\lambda(u^2 - z^2)$, where u is the distance from an axis drawn through O perpendicular to the conducting plane.

218. A distribution of matter M consists of two portions M_1, in a homogeneous medium of inductivity μ_1, and M_2, in a homogeneous medium of inductivity μ_2 surrounding the other medium and reaching to infinity. An equipotential closed surface S_1 surrounds M_1, excludes M_2, and lies wholly in the first medium, a second closed equipotential surface S_2 surrounds M_1, excludes M_2, and lies wholly in the second medium. Prove that if r is the distance from a fixed point O, if normals are drawn outward on S_2 and inward on S_1, and if $d\tau_1$ and $d\tau_2$ are elements of space within S_1 and without S_2 respectively,

$$\mu_1 \int\int \frac{D_n V}{r} dS_1 = 4\pi \int\int\int \frac{\rho\, d\tau_1}{r},$$

and $$4\pi\mu_2 V_O = -\mu_2 \int\int \frac{D_n V}{r} dS_2 + 4\pi \int\int\int \frac{\rho}{r} d\tau_2$$

if O is without S_2, and

$$-\mu_2 \int\int \frac{D_n V}{r} dS_2 + 4\pi \int\int\int \frac{\rho\, d\tau_2}{r} = 4\pi\mu_2 V_{S_2},$$

and $$4\pi\mu_1(V_O - V_{S_1}) = -\mu_1\iint\frac{D_n V}{r}\,dS_1 + 4\pi\iiint\frac{\rho}{r}\,d\tau_1,$$
if O is within S_1.

Show from these equations that if S, the surface of separation of the two media, is equipotential, $\iiint_\infty \frac{\rho\,d\tau}{r}$ is equal to $\mu_2 V_O$ if O is without S, and to $\mu_1 V_O + (\mu_2 - \mu_1)V_S$ if O is within S. Give physical interpretations to these last results. How is the force at any outside point affected by the substitution of one homogeneous dielectric for another in the whole region bounded by S?

219. The open surface S is a surface of zero potential due to a distribution M_1 in an infinite homogeneous medium of inductivity μ_1 on the right of S, and to a distribution M_2 in an infinite homogeneous medium of inductivity μ_2 on the left of S. S is the common boundary of the two media. Show that if r is the distance from a fixed point O, $\iiint_\infty \frac{\rho\,d\tau}{r} = \mu_1 V$ or $\mu_2 V$, according as O is to the right or to the left of S.

220. The function W so vanishes at infinity that $r^2 D_r W$, where r is the distance from any finite point, is not infinite. The normal derivative of W is given at every point of an infinite plane. Prove that if W is harmonic everywhere in the space on one side of the plane, it is determined in that region. Prove also that if W is harmonic in the region on one side of the plane except at the given points $P_1, P_2, P_3, \cdots, P_n$, at each of which it becomes infinite in such a manner that, if r_k is the distance from P_k, and if m_k is a constant belonging to this point, $W - \frac{m_k}{r_k}$ is harmonic at P_k, W is determined in the region in question.

221. Two homogeneous media of inductivities μ_1 and μ_2 have a plane surface of separation but are otherwise unbounded. In the first medium at a point P at a distance a from the

common surface S of the two media is a charge $\epsilon \equiv \mu_1 e$. At Q, any point on S, the force due to this charge has the normal component $ea/(PQ)^3$, or δ, pointing into the second medium. If N_1 and N_2 are the normal components of the whole force at Q pointing into the two media, and if σ' is the apparent density of the surface charge on the plane at Q,

$$N_1 = 2\pi\sigma' - \delta,\ N_2 = 2\pi\sigma' + \delta,$$

and $$\mu_1 N_1 + \mu_2 N_2 = 0,\ N_1 + N_2 = +4\pi\sigma';$$

whence $$N_1 = \delta\,[(\mu_1 - \mu_2)/(\mu_1 + \mu_2) - 1],$$

and $$N_2 = 2\,\mu_1\delta/(\mu_1 + \mu_2).$$

Prove that N_1 might be caused by an apparent charge $(\mu_1 - \mu_2)\,e/(\mu_1 + \mu_2)$ at P', the image of P in the plane, together with an apparent charge e at P and that N_2 might be due to an apparent charge $2\,\mu_1 e/(\mu_1 + \mu_2)$ at P. Hence show by the aid of the theorem stated in the last problem that the potential functions due to these apparent charges are identical (one in the first medium, the other in the second) with the values of the actual potential function in the case described in this problem. The charge at P is urged towards the dielectric with the force $\dfrac{e^2\mu_1}{4\,a^2}\cdot\dfrac{\mu_2 - \mu_1}{\mu_2 + \mu_1}$.

222. Using the notation of Section 62, let the plate A of a spherical condenser be charged with m units of positive electricity and separated from the plate B, which is put to earth, by a spherical shell of radii r and r_i made up of a given dielectric. Let us first ask ourselves what the effect of the dielectric would be if it consisted of extremely thin concentric conducting spherical shells separated by extremely thin insulating spaces. It is evident that in this case we should have a quantity $-m$ of electricity induced on the inside of the innermost shell, a quantity $+m$ on the outside of this shell, a quantity $-m$ on the inner surface of the next shell, a

quantity $+m$ on the outside of this shell, and so on. If there were n such shells in the dielectric layer, and $n+1$ spaces, and if δ were the distance from the inner surface of one shell to the inner surface of the next, and $\lambda\delta$ the thickness of each shell, the value, at the centre of A, of the potential function due to the charges on these shells would be

$$V_A' = m\left[\frac{1}{r+\delta} - \frac{1}{r-\lambda\delta+\delta} + \frac{1}{r+2\,\delta} - \frac{1}{r-\lambda\delta+2\,\delta} \right.$$
$$\left. + \cdots + \frac{1}{r+n\delta} - \frac{1}{r-\lambda\delta+n\delta}\right]$$
$$= -\,m\lambda\delta\left[\frac{1}{(r+\delta)\,(r-\lambda\delta+\delta)} + \frac{1}{(r+2\,\delta)\,(r-\lambda\delta+2\,\delta)} + \cdots\right].$$

This quantity lies between

$$G = -\,m\lambda\delta\sum_{k=1}^{k=n}\frac{1}{(r+k\delta)^2}, \text{ and } H = -\,m\lambda\delta\sum_{k=0}^{k=n-1}\frac{1}{(r+k\delta)^2};$$

but these differ from each other by less than $\epsilon \equiv m\lambda\delta\,\dfrac{r_i^2 - r^2}{r^2 r_i^2}$, so that $-\,m\lambda\displaystyle\int_r^{r_i-(1-\lambda)\delta}\frac{dx}{x^2}$, which is easily seen to lie between G and H, differs from V_A' by less than ϵ. If, then, δ is very small in comparison with r and r_i, V_A' differs from $m\lambda\left(\dfrac{1}{r_i} - \dfrac{1}{r}\right)$ by an exceedingly small fraction of its own value.

This shows that the effect, at the centre of A, of such a system of conducting shells as we have imagined would be practically the same as if a charge $-\,m\lambda$ were given to the inner surface of the dielectric, and a charge $+\,m\lambda$ to its outer surface, while the charges on the surfaces of the thin shells within the mass of the dielectric were taken away. That is, the value of the potential function in A would be

$$m(1-\lambda)\left(\frac{1}{r} - \frac{1}{r'}\right) \text{ instead of } m\left(\frac{1}{r} - \frac{1}{r'}\right).$$

Such a system of shells introduced into what we have hitherto supposed to be the electrically inert insulating matter between the two parts of a spherical condenser would increase the capacity of the condenser in the ratio of 1 to $1-\lambda$. It is to be noticed that λ is a proper fraction: $\lambda = 0$ and $\lambda = 1$ would correspond respectively to a perfect insulator and to a perfect conductor.

If the coatings of a parallel plate air condenser be in the planes $x = 0$, $x = a$, and if the first have a uniform superficial charge of density $-\sigma$ and be kept at potential zero, the potential function in the air between the plates is evidently $4\pi\sigma x$. Show that if a number of plane plates of metal of small thickness $\lambda\delta$ be uniformly distributed between the coatings parallel to the yz plane so as to be separated from each other by air spaces of thickness $(1-\lambda)\delta$, the capacity of the condenser will be increased in the ratio of μ to 1, where $\mu = 1/(1-\lambda)$. Show also that if δ be made infinitesimal and λ a function of x, we have between the coatings in the limit, $\mu D_x V = 4\pi\sigma$, or $D_x(\mu D_x V) = 0$, the differential equation which V would satisfy in a real dielectric of inductivity varying with x. Treat again, on the assumption that λ varies with r, the case of the spherical condenser considered above.

223. The potential function V due to an electrical or magnetic distribution in an inductive medium, may be computed according to the Newtonian Law by taking into account both the intrinsic and the induced charges. If ρ_0 and σ_0 are the intrinsic volume and surface densities, and if the integrations extend all over the space where ρ_0 and σ_0 are different from zero, the potential energy of the distribution is usually written

$$\tfrac{1}{2}\int\int\int V\rho_0\,d\tau + \tfrac{1}{2}\int\int V\sigma_0\,dS,$$

or
$$\frac{1}{8\pi}\int\int_{\infty}\int \mu\{(D_xV)^2 + (D_yV)^2 + (D_zV)^2\}\,d\tau.$$

Why should not the *apparent* volume and surface densities be used in finding the energy by the equation

$$E = \tfrac{1}{2}\iiint V\rho\, d\tau + \tfrac{1}{2}\iint V\sigma\, dS?$$

Answer this question fully, using an illustrative numerical example to explain your assertions.

Assuming that the energy of an electrostatic field would be mathematically accounted for on the supposition that every volume element of space at which the intensity of the field is F contributes $F^2/8\pi$ times its volume to the whole amount, show that if a tube of force be cut into cells by a set of equipotential surfaces drawn at equal potential intervals, these cells contain equal amounts of energy. Show how to divide all space up into unit energy cells. Discuss the mechanical action on a charged conductor in an electric field on the assumption that there is tension along the Faraday tubes which abut on the conductor, such that the normal pull on the conductor per square centimetre of its surface is $F^2/8\pi$. Discuss the pressure at right angles to the Faraday tubes in a dielectric.

224. The space between two concentric spherical surfaces, the radii of which are a and b and which are kept at potentials A and B, is filled with a heterogeneous dielectric, the inductivity of which varies as the nth power of the distance from their common centre. Show that the potential function at any point between the surfaces is

$$(Aa^{n+1} - Bb^{n+1})/(a^{n+1} - b^{n+1}) - a^{n+1}b^{n+1}(A - B)/r^{n+1}(a^{n+1} - b^{n+1}).$$

225. A condenser is formed of two concentric spherical conducting surfaces separated by a dielectric. This dielectric consists of three shells bounded by spherical surfaces of radii r_1, r_2, r_3, and r_4, concentric with the conductors. The inductivities of the inner and outer shells are equal to μ_1, and that

of the intermediate shell is μ_2. Show that if C is the capacity of the condenser,

$$\frac{1}{C}=\left(\frac{1}{r_1}-\frac{1}{r_2}+\frac{1}{r_3}-\frac{1}{r_4}\right)\frac{1}{\mu_1}+\left(\frac{1}{r_2}-\frac{1}{r_3}\right)\frac{1}{\mu_2}.$$

226. The plates of a condenser are two confocal prolate spheroids and the inductivity of the dielectric is A/p, where p is the distance of any point from the axis. Prove that the capacity of the condenser is

$$\pi A/[\log(a_1+b_1)-\log(a+b)],$$

where a, b and a_1, b_1 are the semiaxes of the generating ellipses.

227. The plates of a condenser are the closed metallic surfaces S_1 and S_2. When S_1 is at potential zero and S_2 at potential V_2, the potential function in the air between them is given by the equation $V=f(x, y, z)$. The tube of force based on a portion (S_1') of S_1 abuts on a portion (S_2') of S_2. If the air in this tube were displaced by a homogeneous dielectric of inductivity μ, and if the charges on S_1' and S_2' were increased in the ratio μ, while the charges on the remainder of S_1 and S_2 were unchanged, would the force at every point be unchanged? Would there be a discontinuity in the surface density of the apparent charge on S_1 at the boundary of S_1'?

228. How many square centimetres of tin foil must be used in making a single parallel plate condenser of one microfarad capacity, if the two sheets of foil are to be separated from each other by paraffined paper the thickness of which is one-fifth of a millimetre, and the specific inductive capacity 2? [72,000 π.] Would the required amount be the same if the condenser were made up of a pile of sheets of foil alternating with paper, the odd sheets forming one terminal and the even sheets the other?

229. Show that the generalized Poisson's Equation,

$$D_x(\mu D_x V) + D_y(\mu D_y V) + D_z(\mu D_z V) = -4\pi\rho,$$

is equivalent to

$$h_\xi h_\eta h_\zeta \left[D_\xi \left(\frac{\mu h_\xi D_\xi V}{h_\eta h_\zeta} \right) + D_\eta \left(\frac{\mu h_\eta D_\eta V}{h_\xi h_\zeta} \right) + D_\zeta \left(\frac{\mu h_\zeta D_\zeta V}{h_\xi h_\eta} \right) \right] = -4\pi\rho,$$

if ξ, η, ζ are any orthogonal curvilinear coördinates.

In the case of spherical coördinates, where $h_r = 1$, $h_\theta = 1/r$, $h_\phi = 1/r \sin\theta$, the equation is

$$\sin^2\theta \cdot D_r(\mu r^2 D_r V) + \sin\theta \cdot D_\theta(\mu \sin\theta D_\theta V) + D_\phi(\mu D_\phi V) = -4\pi\rho r^2 \sin^2\theta,$$

and, in columnar coördinates, where $h_r = 1$, $h_\theta = 1/r$, $h_z = 1$, it is $r \cdot D_r(\mu r D_r V) + D_\theta(\mu D_\theta V) + r^2 \cdot D_z(\mu D_z V) = -4\pi\rho r^2$.

230. Show that if the poles of a battery, made up of a given number of equal cells, are to be connected by a resistance R greater than the sum of the resistances of all the cells, the greatest current will traverse R when the cells are joined up in series; but that if R is very small, the cells should be joined up in multiple arc. If R is such that by arranging the cells in a certain number of parallel rows and joining up the numbers of each row in series, the resistance of the whole battery can be made equal to R, this arrangement will give the maximum current.

231. A battery is joined up in simple circuit with a resistance R and a galvanometer of resistance G. After the deflection of the galvanometer has been noted, an additional wire (or shunt) of resistance S is placed across the poles of the battery, and the resistance R is decreased (to r) until the galvanometer deflection is the same as before. Assuming that the electromotive

force of the battery remains constant, show that the resistance of the battery is $\frac{S(R-r)}{G+r}$. [Thomson.]

232. Using the potential function $V = c \log r + d$, where r is the distance from a fixed axis, show that the resistance of a conductor bounded by two concentric circular cylindrical surfaces of radii a and b, and by two planes, distant h from each other, perpendicular to the axis of the cylindrical surfaces, is

$$\frac{\log\left(\frac{b}{a}\right)}{2\pi hk}.$$

Apply the result to the problem of finding the resistance of the liquid in a cylindrical galvanic element.

233. Using the potential function, $V = c \log(r_1/r_2)$, where r_1 and r_2 are the distances from two parallel fixed axes, show how to find (see Fig. 59 and Problem 207) the resistance of a conductor bounded by two parallel planes and by two somewhat eccentric circular cylindrical surfaces which cut the planes orthogonally. In the case of an element in which the zinc electrode is a cylindrical rod and the copper electrode a cylindrical shell surrounding it, is the resistance of the liquid greater or less when the zinc is eccentric to the copper shell than when it is concentric with it?

234. If two points, A and B, of a network of conductors which are carrying steady currents, be connected by an extra conductor W, A and B are said to be at the same potential if no current passes through W. A is said to be at a higher potential than B if a current tends to pass through W from A to B. In this case the difference of potential between A and B is defined to be the electromotive force (in volts) of a galvanic cell which introduced into W with its positive pole towards A would just prevent any current from passing through W.

Three cells of electromotive force 2 volts, 1 volt, and 1 volt respectively, and internal resistances of 1 ohm, 2 ohms, and 4 ohms are joined up in series with a resistance of 1 ohm. Show that the potential differences between the terminals of the separate cells are $+\frac{3}{2}$, 0, and -1 respectively. If the external resistance were 9 ohms, the corresponding potential differences would be $+\frac{7}{4}$, $+\frac{1}{2}$, 0.

235. The terminals of a compound condenser formed of three simple condensers, of capacity 2 microfarads, 3 microfarads, and 6 microfarads respectively, joined up in series, touch the ends of a linear conductor of 22 ohms resistance through which a current of 3 amperes is flowing. What are the charges on the single condensers? Show that if without loss of the charges the condensers be disconnected and joined up in parallel with their positively charged plates in connection, the difference of potential between the terminals of the new compound condenser will be 18 volts. What charge will each of the simple condensers have? [66; 36, 54, 108.]

236. Prove that if a condenser of capacity k farads be charged to potential Q_0/k and then discharged through a large non-inductive resistance, r ohms, the charge Q of the condenser t seconds after the beginning of the discharge is $Q_0 \cdot e^{\frac{-t}{kr}}$; and show that not one ten-thousandth part of the original charge remains after $10\,kr$ seconds.

Show also that the energy that has been expended up to the time t in heating the wire is

$$\frac{Q_0^2}{2\,k}(1 - e^{-\frac{2t}{kr}}) \text{ joules.}$$

237. The terminals of a condenser of k farads capacity are attached permanently to the poles of a constant battery of electromotive force E volts by leads of large resistance, r ohms. After the condenser has become fully charged its

terminals are suddenly connected together without removing the battery by a conductor of large resistance, R ohms. Assuming that the solution of a differential equation of the form $D_t y + ay = b$ is $y = b/a + Be^{-at}$, show that at the time t the charge on one of the condenser plates is

$$Q = \frac{Ek}{R + r}(R + re^{-qt}),$$

where $$q = (R + r)/rRk.$$

238. A galvanic battery is composed of two galvanic cells, the electromotive forces of which are e_1 and e_2 and the internal resistances b_1 and b_2, joined up in multiple arc. The poles of the battery are connected by an external resistance of r ohms. Show that if C_1 and C_2 are the strengths of the currents flowing through the cells,

$$C_1 = [e_1 b_2 + r(e_1 - e_2)] \div [b_1 b_2 + r(b_1 + b_2)],$$
$$C_2 = [e_2 b_1 + r(e_2 - e_1)] \div [b_1 b_2 + r(b_1 + b_2)].$$

239. A galvanometer of 9 ohms resistance is to be furnished with two shunts, such that when the first alone is used $\frac{1}{10}$ of the current shall pass through the instrument, and that when both are used in parallel, 29/30 of the current shall pass through them. Prove that the resistance of the second shunt must be 9/20.

240. A storage battery is used to send a current through a cluster of incandescent lamps arranged in multiple arc. The resistance of each lamp when hot is 100 ohms. When 10 lamps are used the current through each is 1 ampere, but when 20 are used this current is only $\frac{11}{12}$ of an ampere. Find the resistance of the battery and its connections and show that the electromotive force of the battery is 110 volts.

241. If a number of cells of different electromotive forces but of equal internal resistances are joined up in multiple arc, the battery thus formed is equivalent, so far as its ability

to send currents through outside resistances is concerned, to a single cell the electromotive force of which is the mean of the electromotive forces of the cells in the battery. Find the resistance of this equivalent cell and show that it would be more "effective" when doing a given amount of external work than the battery. How much work is done in the battery per second when the external circuit is broken?

242. A certain uniform cable 50 kilometres long has, when in good condition, a resistance of 450 ohms. The operator at one end finds that the resistance is 270 ohms or 350 ohms according as the other end is grounded or insulated. Supposing the ground connections at the two stations to be good, so that the resistance of the earth is negligible, and assuming that there is a single fault in the cable, show that this fault is 16.67 kilometres from the first station and that its resistance is 200 ohms.

243. A cable 500 kilometres long with stations *A* and *B* at its extremities has a single fault, but is not so much injured that signals cannot be sent through it. With cable insulated at *B*, the operator at *A* grounds one terminal of a large battery and attaches the other terminal to the cable. After this has been done the operators find that the difference of potential between the cable and the ground is 200 volts at *A* and 40 volts at *B*. The cable at *A* is then insulated, and one terminal of a large battery at *B* is grounded while the other is attached to the cable. The difference of potential between the cable and the ground is then 300 volts at *B* and 40 volts at *A*. Show that the fault has a resistance equivalent to that of 47.62 kilometres of cable and is at 190.5 kilometres from *A*. Explain some way of measuring the potential differences in this case.

244. "In a network *PA*, *PB*, *PC*, *PD*, *AB*, *BC*, *CD*, *DA*, the resistances are

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \gamma+\delta,\ \delta+\alpha,\ \alpha+\beta,\ \beta+\gamma$$

respectively. Show that if AD contains a battery of electromotive force E, the current in BC is

$$\frac{\lambda(\alpha\beta + \gamma\delta) E}{2\lambda^2\mu + (\beta\delta - \alpha\gamma)^2},$$

where $$\lambda \equiv \alpha + \beta + \gamma + \delta,$$

and $$\mu = \beta\gamma + \gamma\alpha + \alpha\beta + \alpha\delta + \beta\delta + \gamma\delta."$$

245. Show that if the edges of a parallelopiped be formed of uniform wire such that the resistances of three conterminous edges are a, b, and c respectively, and if a current enters at one angle and leaves at the opposite angle, the resistance of the network is $\frac{1}{4}[(a + b + c) + abc/(ab + bc + ca)]$.

246. (*a*) A tetrahedral framework is made of uniform wire, opposite edges being equal and of lengths a, b, c. If a current enters and leaves the framework at the ends of an edge of length a, the strengths of the currents in the pairs of edges of length a are in the ratio

$$b(a + c) + c(a + b) : b(a + c) - c(a + b).$$

[Jesus College.]

(*b*) Show that the resistance of the whole framework is that of a length of the wire equal to $\frac{1}{2}[ab/(a + c) + ac/(a + b)]$. [St. John's College.]

247. Show that if n telegraph poles, each of resistance R, be joined in pairs, each to all the others, with wires of resistance r, and if an electromotive force E be inserted in one of the wires, the current in that wire is $E\{R(n - 2) + r\}/r(nR + r)$.

248. An electric distributing conductor 6 miles long gives out continuously 50 amperes of current per mile of its length. The end of the conductor remote from the generator is insulated, while the nearer end is kept at 1000 volts potential. Show that if the resistance per mile of the conductor is

1 ohm, the voltage at a point on the line x miles from the generator is $v = 50\,x\left(\frac{x}{2} - 6\right) + 1000$. Find the rate at which a given portion of the line is delivering power.

249. A Wheatstone's bridge in proper adjustment consists of four conductors, AB, BC, CD, DA, which have respectively the resistances p, q, s, and r. The galvanometer is connected with A and C and the battery with B and D. The electromotive force of the battery is E, and the resistance of the battery with its connecting wires is b. Prove that the heat developed per unit time in the conductor AB is the equivalent of the energy $\dfrac{E^2qrs}{[b(s+r)+r(q+s)]^2}$.

250. A generator of constant electromotive force E and of constant internal resistance B is used to charge a storage battery which now has an electromotive force e and an internal resistance b. Show that if the poles of the storage battery be connected by a conductor of resistance r, a current

$$C' = (Be + bE) \div [(B + b)\,r + Bb]$$

will go through this conductor.

251. The conductors AB, BC, CD, and DA have the resistances p, q, r, and s respectively. A is connected with C by a battery of internal resistance b and electromotive force e. B is connected with D by a battery of internal resistance b' and electromotive force e'. Prove that if the current in AC is zero,

$$e\{b'(p+q+r+s) + (p+s)(q+r)\} + e'(pr - qs) = 0.$$

252. A conductor of given dimensions made of given material has two given portions S_1 and S_2 of its surface kept at constant potentials while the rest of its surface is a current surface. Show that if V is the potential function within the conductor, when S_1 is kept at potential C_1 and S_2 at potential C_2, and

if V' is the potential function, when S_1 is kept at C_1' and S_2 at C_2',

$$V' = \frac{C_1' - C_2'}{C_1 - C_2} V + \frac{C_2'C_1 - C_2C_1'}{C_1 - C_2}.$$

253. One end (O) of a straight wire of radius a and length l is kept at potential V_0, and the other end (Q) at potential V_1. The specific conductivity of the wire is κ and its resistance per unit length is w, so that the reciprocal of w is equal to $\pi a^2 \kappa$. The wire is surrounded by an insulating sheath, the outside of which is in contact with sea water at potential zero. The rate of leakage per unit length of the wire or cable through the sheath at a place where the potential of the wire is V is $2\pi a \lambda V$. The reciprocal of $2\pi a \lambda$ is denoted by W and is called the "insulation resistance" of the cable per unit length. The rate of flow of electricity into a portion of the cable of length Δx, included between two right sections, the nearer of which is distant x from O and is at potential V, is $-\kappa \pi a^2 D_x V$. The rate of flow of electricity out of this element through the sheath and from the farther end is $-\kappa \pi a^2 (D_x V + \Delta_x D_x V) + 2\pi a \lambda V \Delta x$. When the current is steady the element neither gains nor loses electricity and $\kappa \pi a^2 \Delta_x D_x V - 2\pi a \lambda V \Delta x = 0$, so that at every point $D_x^2 V - \beta^2 V = 0$, where $\beta^2 = w/W$. The general solution of this equation is of the form $V = Ae^{\beta x} + Be^{-\beta x}$, and if we determine A and B so that $V = V_0$ when $x = 0$, and $V = V_1$ when $x = l$, we get $V = [V_1 \sinh(\beta x) + V_0 \sinh \beta(l - x)]/\sinh(\beta l)$.

Show that if the current which enters the cable at O is I_0 and that which leaves it at Q is I_1, and if I denote the current in the core at a point at a distance x from O,

$$I = [V_0 \cosh \beta(l - x) - V_1 \cosh(\beta x)]/[\sqrt{\omega W} \sinh(\beta l)]$$

$$= I_0 [V_0 \cosh \beta(l - x) - V_1 \cosh(\beta x)]/[V_0 \cosh(\beta l) - V_1],$$

$$I_1 = [V_0 - V_1 \cosh(\beta l)]/[\sqrt{\omega W} \sinh(\beta l)]$$

$$= I_0 [V_0 - V_1 \cosh(\beta l)]/[V_0 \cosh(\beta l) - V_1].$$

Show also that if the end of the cable at Q be insulated and left to itself, $V_1 = \frac{V_0}{\cosh(\beta l)}$, but if it be put to earth, $V = V_0 \sinh \beta(l - x) / \sinh(\beta l)$. If in this latter case the cable were infinitely long, we should have $V = V_0 e^{-\beta x}$ and $I = I_0 e^{-\beta x} = V_0 e^{-\beta x} / \sqrt{\omega W}$.

The whole core resistance of a certain cable 1000 miles long is 2000 ohms. When one terminal of a battery (the other terminal of which is put to earth) is attached to one end of the cable and the other end of the cable is grounded, the current at the sending end is to the current at the receiving end as 1.1276 to 1. Show that the insulation resistance of the cable per mile is 8 megohms. In the Atlantic cable of 1889, $w = 1.54$ ohms per kilometre, and $W = 9{,}085{,}000{,}000$ ohms per kilometre.

254. The conduction resistance of a certain cable 1000 miles long is 10 ohms per mile, whilst the insulation resistance is 10 megohms: if the sending end be at a given potential and the receiving end to earth, find the whole charge of the cable when a steady current passes through it. Show that if the cable have a leakage fault at the middle point the resistance of which is equal to that of a length of a miles of the cable, the strength of a steady current at the receiving end will be lowered in the ratio $1 : 1 + \frac{500}{a} \cdot \frac{e-1}{e+1}$. [M. T.]

255. Prove that if any finite set of algebraic operations be performed upon the complex variable $z = x + yi$ taken as a whole, and if the result $[w = f(z)]$ be written in the form $\phi(x, y) + i \cdot \psi(x, y)$, where ϕ and ψ, which are said to be conjugate to each other, are real functions of x and y:

(*a*) Both ϕ and ψ satisfy Laplace's Equation.

(*b*) $D_x\phi = D_y\psi$ and $D_y\phi = -D_x\psi$.

(*c*) At any point P, the derivative of ϕ taken in any direction PQ in the plane xy is equal to the derivative of ψ taken in a direction PR at right angles to PQ, and such that the angle QPR corresponds to a counter-clockwise rotation.

(*d*) The equations $\phi(x, y) = c$, $\psi(x, y) = c'$ represent two families of curves which cut each other orthogonally.

256. Prove that:

(*a*) If ϕ and ψ are any two conjugate functions of x and y, that is, if $\phi + i\psi$ is a function of the complex variable $x + yi$, taken as a whole, then, conversely, x and y are two conjugate functions of ϕ and ψ.

(*b*) If ϕ and ψ are any two conjugate functions of x and y, and if α and β are any two other conjugate functions of x and y, and if for x and y in the expressions for ϕ and ψ we substitute the expressions for α and β, we shall get two new conjugate functions of x and y.

(*c*) If ϕ_1 and ψ_1, ϕ_2 and ψ_2 are any two pairs of conjugate functions, $\phi_1 \pm \phi_2$ and $\psi_1 \pm \psi_2$ are conjugate functions of x and y.

257. Prove that in any case of steady uniplanar flow of electricity — that is, flow which at every point is parallel to a given plane, and of such a character that its intensity and direction are the same at all the points of any line drawn perpendicular to the given plane — there exists a function conjugate to the potential function. This function is called the "flow function."

258. Show by the ordinary rules for treating imaginary quantities that, if $z = x + yi$, z^2, $\sqrt{z}$, $\log z$ will yield respectively the following pairs of conjugate functions: $A(x^2 - y^2)$, $-2Axy$; $Ar^{\frac{1}{2}} \cos\frac{\theta}{2}$, $Ar^{\frac{1}{2}} \sin\frac{\theta}{2}$; $A \log r$, $A\theta$; where $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\frac{y}{x}$. State some problems of steady flow within conductors which these conjugate functions will help to solve.

259. Show that, with certain broad limitations, either one (say ϕ) of any pair (ϕ, ψ) of conjugate functions of x and y may be taken as the potential function in empty space due to an electrostatic distribution the density of which is a function of x and y only, and which, therefore, must be constant at all points on any indefinitely extended line drawn perpendicular to the

plane of xy. Show also that in the case of the same distribution the other function ψ will be constant along any line of force.

260. Show that either one (say ϕ) of any pair (ϕ, ψ) of conjugate functions of x and y may be taken as the potential function inside a conductor which carries a steady current flowing at every point in a direction parallel to the plane of xy, and the same in intensity and direction at all points of any line drawn perpendicular to this plane. Show that in this case the other function ψ will be constant along any line of flow, and that the two equations $\phi = c$, $\psi = c'$ represent respectively, if c and c' are parameters, cylindrical equipotential surfaces and cylindrical surfaces of flow. If ds is the element of any curve AB in the plane xy, and if $D_n\phi$ is the derivative of ϕ taken in the direction of the normal to ds which points towards the right as one goes along the curve from A to B, the integral $-k\int_A^B D_n\phi \cdot ds$ gives the amount of positive electricity which crosses per unit of time from left to right so much of a right cylindrical surface erected on AB as is enclosed by two planes parallel to the plane of xy and at the unit distance from each other. Since $D_n\phi = D_s\psi$, the integral just considered is equal to $-k(\psi_B - \psi_A)$, and $-k$ times the difference between the values of ψ on two right cylindrical surfaces of flow gives the amount of flow across the unit height of so much of any cylindrical surface which cuts the plane of xy at right angles as is included between the given surfaces of flow.

261. Prove that:

(*a*) If $r_1, r_2, r_3, \cdots, r_n$ are the lengths of the radii vectores drawn from any point P to any n parallel axes, and if $\theta_1, \theta_2, \theta_3, \cdots, \theta_n$ are the angles which these radii vectores make with a fixed line in the plane of xy which is perpendicular to the axes,

$$\phi \equiv A_1 \log r_1 + A_2 \log r_2 + A_3 \log r_3 + \cdots + A_n \log r_n,$$
$$\psi \equiv A_1\theta_1 + A_2\theta_2 + A_3\theta_3 + \cdots + A_n\theta_n$$

are conjugate solutions of Laplace's Equation.

(*b*) The equation $\psi = c'$ represents for each value of c' a cylindrical surface which passes through all the axes.

(*c*) For very large values of c, the equation $\phi = c$ represents as many closed cylindrical surfaces, each surrounding one of the axes, as there are positive terms in the expression for ϕ.

(*d*) For very large negative values of c, the equation $\phi = c$ represents as many closed cylindrical surfaces, each surrounding one of the axes, as there are negative terms in ϕ.

(*e*) If $\Sigma A = 0$, no one of the cylindrical surfaces $\psi = c'$ ends at infinity.

(*f*) The value of $\int D_n\phi \cdot ds$ taken around any closed curve in the plane xy which surrounds the jth axis and no other is equal to the change made in ψ by going around the curve, and this is $2\pi A_j$.

(*g*) However the axes may be distributed and whatever values may be assigned to the A's, ϕ represents the potential function corresponding to a uniplanar flow of electricity* within the substance of an infinite conducting lamina, either thick or thin, when cylindrical holes, on the curved surface of each one of which ϕ is constant, are cut through the lamina so as to remove all the axes, and if the curved surfaces of these holes are kept at potentials equal to the values of ϕ on them. This is practically the case of a very large thin sheet of metal touched at certain points by the ends of wires connected with the poles of batteries.

(*h*) If in the value of ϕ there is an even number $(2m)$ of terms, half of which are positive and half negative, and if, moreover, all the A's are numerically equal, we have the case in which m similar pieces of wire connected with the positive pole of a battery touch a thin sheet of metal in m places, and m similar pieces of wire connected with the negative pole of the battery touch the metallic sheet in m other places. In this case, if P_1 and P_2 are any two points in the metal, the resistance of so much of the sheet as lies between the equipotential surfaces on which P_1 and P_2 lie is $\frac{\phi_{P_2} - \phi_{P_1}}{2\pi Ank\delta}$, when δ is the thickness of the lamina, and k its specific conductivity.

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* See papers by Foster and Lodge in the *Philosophical Magazine* for 1875 and 1876.

(i) If ϕ consists of two terms the coefficients of which are numerically equal but opposite in sign, we have the case of a thin sheet of metal touched at two points by the two poles of a battery. Here the curves in the plane xy, for which ψ is constant, are circles (Fig. 59) the centres of which are on the line which bisects at right angles the line which gives the points where the battery electrodes touch the sheet.

Show that this value of ϕ enables us to find the resistance of a thin circular disc touched at two points on its circumference by the poles of a battery, and hence, by superposition, the resistance of such a disc touched by any number of pairs of battery poles at different places on the circumference. State other problems which an inspection of Fig. 59 shows can be solved by the aid of the value of ϕ.

(j) If ϕ is made up of an infinite number of terms with coefficients all numerically equal, but alternately positive and negative, and if the corresponding axes cut the plane of xy in a straight line so that the distance between any axis and the next is b, certain of the lines of force in the plane of xy will be straight lines which cut at right angles the line on which the traces of the axes lie. Show that by aid of this ϕ we can find the resistance of a lamina of breadth b, and of infinite length when touched at two points opposite each other, one on one edge, and the other on the other. Draw from general knowledge a diagram which shall give the shape of the lines of flow and the equipotential lines in such a lamina.

262. (a) Show that if in a thin conducting plate of indefinite extent there are two sources and a sink, each of strength

numerically equal to m, situated respectively at points A, B, C, which lie in order upon a straight line, one of the lines of flow consists in part of a circumference of radius $\sqrt{CA \cdot CB}$ drawn around C as a centre, so that the flow inside the circumference would be unchanged if the part of the plate outside it were cut away. In other words, if a circumference be drawn in a thin conducting plane plate of indefinite extent, the "image" in this circumference of a source, of strength m, situated at a point P in the plane, is made up of a sink, of strength m, at the centre of the circle, and a source of the same strength at Q, the inverse point of P with respect to the circumference.

Show that if a sink be regarded as a negative source, and if, inside a circumference drawn in a thin plane conducting plate of indefinite extent, there are sources at the points A_1, A_2, A_3, $\cdots$, A_k, of strengths algebraically equal to m_1, m_2, m_3, $\cdots$, m_k respectively, and sources of strengths algebraically equal to $-m_1$, $-m_2$, $-m_3$, $\cdots$, $-m_k$, at the corresponding inverse points, then, if $m_1 + m_2 + m_3 + \cdots + m_k = 0$, there is no flow of electricity across the circumference.

If at a fixed point P in a thin plane plate (Fig. 127) there is a sink of strength numerically equal to m, and at another point P' in the plate an equal source, and if P' be made to approach P as a limit and the product $m \cdot \overline{PP'}$ be kept always equal to a given constant μ, we have as a limit a "plane doublet"* of strength μ, the axis of which is PX, the limiting position of the straight line drawn *from P to P'*. We shall find it convenient to represent sources and sinks respectively by black and unshaded circles, and doublets by circles half black and half unshaded. The black portion

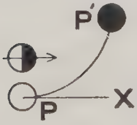

FIG. 127.

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* Kirchhoff, *Pogg. Ann.*, 1845, p. 497. W. R. Smith, *Proc. Ed. Roy. Soc.*, 1869–70. Foster and Lodge, *Phil. Mag.*, 1875. Minchin's *Uniplanar Kinematics*, p. 213. Peirce, *Proc. Am. Acad. of Arts and Sciences*, 1891.

of a doublet circle indicates the directions in which there is a flow *away from* the point where the doublet is situated; the unshaded portion indicates the directions from which there is a flow *towards* this point. The axis of a doublet bisects both the black and unshaded portions of the doublet circle. Show that if P be used as origin and PX as axis of abscissas, the velocity potential function due to the doublet is $\phi = -\frac{\mu x}{x^2 + y^2}$, and the flow function is $\psi = \frac{\mu y}{x^2 + y^2}$. If $x + yi = z$, these are respectively the real part and the real factor of the imaginary part of the function $\frac{-\mu}{z}$. The equipotential lines and the lines of flow are circles (see Fig. 128) touching the axes of y and x respectively at the origin.

A "plane quadruplet" is formed of two equal and opposite plane doublets in the same manner that a doublet is formed out of a source and an equal sink. An "octuplet" is formed in a similar way of two equal and opposite quadruplets, and so on. We may use the word "motor" to denote in general a source, a sink, a doublet, a quadruplet, or any other combination of sources or sinks at a single point.

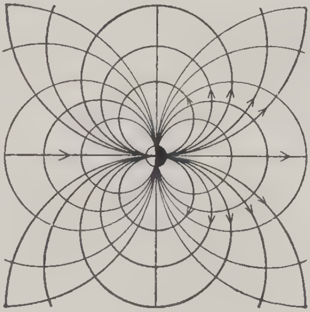

FIG. 128.

(*b*) The upper circle in Fig. 130 shows the plane quadruplet formed by combining the two plane doublets indicated in the lower part of this diagram. Show that the flow function due to a quadruplet of this kind at the origin is $-2kxy/(x^2 + y^2)^2$, while the flow function due to such a quadruplet as that shown in Fig. 131 will be $k(x^2 - y^2)/(x^2 + y^2)^2$. One of these quadruplets is evidently equivalent to the other turned through 45°. Find the flow function due to an octuplet of the kind shown in Fig. 132 at the origin.

(*c*) Show that the lines of flow due to a plane doublet may be regarded as the lines of force due to a columnar magnet of infinitely small cross-section.

(*d*) Show that the functions

$$\log z, \frac{1}{z}, -\frac{1}{z^2}, \frac{2}{z^3}, -\frac{6}{z^4}, \cdots,$$

each of which is the derivative with respect to z of the one which precedes it, yield a series of pairs of conjugate functions which represent in order the velocity potential functions and the flow functions due to a source at the origin, to a plane doublet at the origin, to a plane quadruplet at the origin, to a plane octuplet at the origin, and so on.

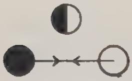

FIG. 129.

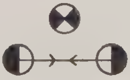

FIG. 130.

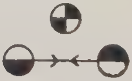

FIG. 131.

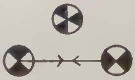

FIG. 132.

(*e*) Show that if two plane doublets L and M exist together at a point O, and if the directions of the two straight lines OA, OB show the directions of the axes of L and M respectively, and the lengths of OA and OB the strengths of L and M on some convenient scale, then the direction of the axis of the resultant of L and M will be given by the direction, and the strength of the resultant by the length, of the diagonal of the parallelogram of which OA and OB are adjacent sides. Plane doublets, then, can be compounded and resolved by compounding and resolving their axes like forces or velocities.

263. If a charge $+ m$ concentrated at a point Q be made to approach on any analytic curve a point charge $- m$ at a fixed point P on the curve, and if as Q approaches P, m is made to increase in such a manner that the product of m and PQ is

always equal to the constant μ, the limiting value of the potential function of the system is said to be due to a *space doublet* of strength μ at the point P, and the axis of the doublet is said to be the limiting position of the secant PQ. Show that if r is the distance of any point P' from P and if θ is the angle between the axis of the doublet and PP', the value at P' of the potential function due to the doublet is $\mu \cos \theta / r^2$.

The force components along and perpendicular to r are $2\mu \cos \theta / r^3$ and $\mu \sin \theta / r^3$. The potential function (Section 69) due to a doublet at the origin with axis coincident with the x axis is $\mu x / r^3$.

The potential function due to a mass $-m$ at the point $(b, 0, 0)$, a mass $+m$ at the point $(b + \delta, 0, 0)$, a mass $-ma/(b + \delta)$ at the point $(a^2/(b + \delta), 0, 0)$, and a mass ma/b at the point $(a^2/b, 0, 0)$, where b and δ are smaller than a, has the value zero on the spherical surface $x^2 + y^2 + z^2 = a^2$. Prove that if, while a and b are constant, δ be made to decrease indefinitely and m to increase in such a manner that their product shall always be equal to the given constant μ, the limiting value of the potential function will be

$$\mu (x - b) / [(x - b)^2 + y^2 + z^2]^{\frac{3}{2}} + a\mu \, [b (x^2 + y^2 + z^2) \\ - a^2 x] / [(bx - a^2)^2 + b^2 (y^2 + z^2)]^{\frac{3}{2}}.$$

If $b = 0$, this expression becomes $\mu x (a^3 - r^3) / a^3 r^3$, where $r^2 = x^2 + y^2 + z^2$. What problem in electrostatics can be solved by the aid of this last function? Is the image of a doublet in a spherical surface another doublet?

264. A straight wire of radius a which forms the core of a cable of length l lies in the axis of x with one end at the origin and the other at the point $(l, 0, 0)$. The whole of the outside of the insulating covering of the cable and the core at the point $(l, 0, 0)$ are kept at potential zero, while the core at the origin is at the potential V_0. Show that if c is the capacity per unit length of the cable considered as a

condenser, k the ratio of the conductivity per unit length of the core to c, and h the rate of loss of electricity by leakage through the insulation per unit length of the cable when the difference of potential of the core and the outside of the cable is unity, the value of the potential function V in the core satisfies the equation

$$D_t V = k \cdot D_x^2 V - \frac{h}{c} V, \text{ and if } V \equiv we^{-ht/c},\ D_t w = k \cdot D_x^2 w \cdot$$

Show also that in the final state, when V satisfies the equation $D_x^2 V = hV/kc$ and is equal to V_0 when $x = 0$, and to zero when $x = l$, the value of V is given by the expression $[V_l \cdot \sinh \beta x + V_0 \cdot \sinh \beta (l - x)]/\sinh \beta l$, where $\beta^2 = h/kc$.

Prove that any quantity of the form $A_s \cdot e^{-\lambda t} \cos (nx - \delta)$, where $\lambda \equiv kn^2$, satisfies the equation $D_t w = k \cdot D_x^2 w$, and that if $\delta \equiv \frac{1}{2}\pi$, $n \equiv s\pi/l$, where s is an integer, and

$$A_s \equiv -2\, ck\pi s \cdot \cos s\pi \cdot /(hl^2 + cks^2\pi^2);$$

the expression

$$w_1 \equiv \sum_{s=1}^{s=\infty} A_s e^{-\lambda t} \sin nx \text{ vanishes when } x = 0 \text{ or } x = l,$$

and, when $t = 0$, is equal to $-\sinh \beta (l - x)/\sinh \beta l$. Hence prove that the expression

$$V = V_0[\sinh \beta (l - x)/\sinh \beta l - w_1 e^{-ht/c}]$$

gives the value of the potential function in the cable, if, when the whole core is at potential zero and the farther end permanently grounded, the point $x = 0$ is suddenly raised to potential V_0 at the time $t = 0$, and kept there. The current (C) at any point is given by the negative of the derivative of the potential function with respect to x, divided by the resistance ρ of the core per unit length, so that

$$\rho C = V_0[\beta \cdot \cosh \beta (l - x)/\sinh \beta l + (s\pi/l)\pi e^{-ht/c} \sum s \cdot A_s e^{-\lambda t} \cos nx].$$

If the insulation is so good that h may be neglected,

$$V = V_0[(l-x)/l - 2 \cdot \cos s\pi \cdot \sum (e^{-\lambda t} \sin nx)/\pi s],$$

and the current is

$$(V_0/\rho l)(1 + 2 \cdot \cos s\pi \cdot \sum e^{-\lambda t} \cos nx).$$

265. The terminals of a battery of electromotive force E_0 volts and internal resistance b ohms are suddenly connected, through a non-inductive conductor of resistance $r - b$ ohms, with the coatings of a condenser of k farads capacity. Show that after t seconds the condenser is charged to potential difference E volts, where $E = E_0(1 - e^{-t/kr}) = E_0 T$, and that the charge on the positive plate is Ek units. If $t = \frac{1}{10} kr$, $T = 0.095$; if $t = \frac{1}{5} kr$, $T = 0.181$; if $t = \frac{1}{2} kr$, $T = 0.393$; if $t = kr$, $T = 0.632$; if $t = 2 kr$, $T = 0.865$; if $t = 3 kr$, $T = 0.950$; if $t = 5 kr$, $T = 0.993$, and if $t = 7 kr$, $T = 0.999$.

Show that if the condenser just mentioned had been leaky, its dielectric having a resistance of only R ohms, the charge on the positive coating after t seconds would have been $\dfrac{E_0 kR}{r+R}(1 - e^{-t(r+R)/krR})$, and the final charge $E_0 kR/(r+R)$.

266. The coatings of a perfect condenser of 2 microfarads capacity which are connected together by a non-inductive resistance R of 2500 ohms are attached to the terminals of a constant battery. After the condenser has become fully charged, a bullet moving at a velocity of v metres per second cuts first one of the battery leads at a point A and, 2 metres farther on in its course, the resistance R at a point B. While the bullet is moving from A to B the condenser loses $1 - 1/e$ of its charge through R. Show that, e being the base of the natural system of logarithms, $v = 400$.

267. If S_1 and S_2, the plates of a condenser separated by a poorly conducting medium of inductivity μ and of conductivity λ, are at potentials V_1 and V_2 respectively, and if V denotes

the potential function in the dielectric, the capacity of the condenser and the strength of the current that flows through the dielectric, when the difference of potential of the plates is unity, are

$$\mu\left[\int D_n V \cdot dS_1\right]/4\pi(V_2 - V_1) \text{ and } \lambda\left[\int D_n V \cdot dS_1\right]/(V_2 - V_1).$$

Show that if the condenser be charged to such a potential that each plate requires Q_0 units of (positive or negative) electricity and then, left to itself, the charge on one of the plates after t seconds is given numerically by the expression $Q = Q_0 e^{-4\pi\lambda t/\mu}$.

268. A Leyden jar loses 0.000001 of its charge per second by conduction through the glass. The specific inductive capacity of the glass is 8. Show that the resistance of a cubic centimetre of the glass is roughly 14×10^{17} ohms, having given that one electrostatic unit of resistance is equivalent to 9×10^{11} ohms. [M. T.]

269. A submarine telegraph cable 1885 miles long is formed of a copper conductor $\dfrac{1}{2e}$ inches in diameter surrounded by a gutta-percha coating $\frac{1}{2}$ inch in diameter. The specific inductive capacity of gutta-percha being 4.2, show that the capacity of the cable is equal to that of a sphere of the same size as the earth. [St. John's College.]

270. The outer coatings of two condensers A and B are put to earth and their inner coatings are connected through a galvanometer the resistance of which is 4000 ohms. The capacity of A is 3 microfarads, that of B is 1 microfarad, and the two condensers are charged to potential 1 volt. The inner coatings of A and B are then put to earth simultaneously through resistances of 1000 and 2000 ohms respectively. Show that the whole amount of electricity which will flow through the galvanometer is one-seventh of the charge of the smaller condenser. [St. John's College.]

271. The outer coatings of two condensers A and B are put to earth and their inner coatings are connected together through a galvanometer of g ohms resistance. The capacities of the condensers are C and c respectively. Both are charged initially to potential V_0 and then have charges Q_0 and q_0. Show that if the inner coatings of the condensers are put to earth simultaneously through non-inductive resistances R and r, and if

$$\lambda \equiv r\,RC,\ \ \lambda' \equiv rRc,\ \ \mu \equiv cr(g+R),\ \ \mu' = CR(g+r),$$

$$m \equiv CcrRg,\ k^2 \equiv 4\,\lambda\lambda' + (\mu - \mu')^2;\ \mu\mu' - \lambda\lambda' = CcrRg(g+r+R),$$

and the charge on A after t seconds will be

$$Q_0 e^{-(\mu+\mu')t/2m}\big[(k+\mu+\mu'-2\,m/CR)\,e^{kt/2m} + (k-\mu-\mu'+2\,m/CR)\,e^{-kt/2m}\big]/2\,k.$$

Show also that the whole quantity of electricity which passes through the galvanometer during the discharge is

$$Q_0(CR - cr)/C(g+r+R).$$

272. Prove that the potential and stream line functions due to electrodes placed at certain points of a spherical current sheet can be deduced directly from the solutions for the plane current sheet which is its stereographic projection. If E_1 and E_2 be two electrodes on a complete spherical sheet, show that the stream lines are small circles through E_1 and E_2 and the equipotential curves small circles the planes of which pass through the line of intersection of the tangent planes at E_1 and E_2.

273. Verify the statement that the value of the potential function at any point P of a solid homogeneous sphere of specific resistance κ, when a current of intensity C flows between two electrodes A and B at opposite ends of a diameter, is

$$\frac{C\kappa}{2\,\pi}\left[\frac{1}{AP} - \frac{1}{BP} - \frac{1}{AB}\cdot\log\frac{AN+AP}{BN+BP}\right],$$

where N is the foot of the perpendicular from P on the diameter AB. [M. T.]

274. The two concentric spherical surfaces which bound a shell are kept at different constant potentials. Prove that if the conductivity of the shell is a function of the distance from its centre, the potential function within it satisfies the equation $D_r(r^2k \cdot D_r V) = 0$. Show that if $w \equiv 1/r$, this is equivalent to the equation given on page 250.

275. Prove that if a quantity of electricity equivalent to Q absolute electromagnetic units be discharged through a ballistic galvanometer which has a suspended system the magnetic moment of which is M, the moment of inertia I, and the reduced complete time of swing T_0,

$$Q = \frac{4\pi I}{GMT_0} \sin \tfrac{1}{2}\alpha = 2\frac{H^{\frac{1}{2}} I^{\frac{1}{2}}}{GM^{\frac{1}{2}}} \sin \tfrac{1}{2}\alpha,$$

where GM is the couple exerted upon the suspended system in its position of equilibrium when a steady current of 1 unit passes through the galvanometer coil.

276. When a bar magnet of magnetic length $2l$ and moment M is placed in Gauss's A position with its centre at a distance d from the centre of a magnetic needle of length 2λ, the needle is deflected through an angle α, such that

$$\frac{2H\tan\alpha}{M} = \frac{d-l}{r_1^{\,3}} + \frac{d-l}{r_2^{\,3}} - \frac{d+l}{r_3^{\,3}} - \frac{d+l}{r_4^{\,3}},$$

where
$$r_1^{\,2} = (d - l - \lambda \sin\alpha)^2 + \lambda^2 \cos^2\alpha,$$
$$r_2^{\,2} = (d - l + \lambda \sin\alpha)^2 + \lambda^2 \cos^2\alpha,$$
$$r_3^{\,2} = (d + l - \lambda \sin\alpha)^2 + \lambda^2 \cos^2\alpha,$$
$$r_4^{\,2} = (d + l + \lambda \sin\alpha)^2 + \lambda^2 \cos^2\alpha.$$

Show that if $l = 4$ centimetres, $d = 40$ centimetres, $\lambda = 0.5$ centimetre, $\alpha = 20^0$, and $H = 0.2$, this formula makes $m = 285.43$, whereas the approximate formula,

$$\frac{M}{H} = \frac{(d^2 - l^2)^2}{2d} \tan\alpha, \text{ yields } m = 285.40.$$

277. A magnetometer is set up with the centre of its needle vertically above a point in the axis of a horizontal metre rod n centimeters from the centre. The rod is perpendicular to the meridian. A homogeneous, short bar magnet is placed in Gauss's A position with its centre first $50 - d$ centimetres from one end of the rod and then $50 - d$ centimetres from the other end, d being greater than n. If the deflections of the magnetometer needle in the two cases are δ_1 and δ_2 respectively, the relative error made by computing M/H by means of the formula

$$d^3(\tan \delta_1 + \tan \delta_2)/4 \text{ is } [(1 + 3e^2)/(1 - e^2)^3] - 1,$$

where $e = n/d$.

278. The track upon which the carriage of the short deflecting magnet slides in an apparatus for determining M/H in Gauss's A position makes an angle θ with the east and west line instead of being exactly perpendicular to the meridian. Show that if the centre of the deflecting magnet is at a distance d from the centre of the needle, and if the deflection changes from δ_1 to $-\delta_2$ when the deflector is turned end for end,

$$\frac{M}{H} = \frac{d^3 \sin \delta_1}{2 \cos (\delta_1 + \theta)} = \frac{d^3 \sin \delta_2}{2 \cos (\delta_2 - \theta)},$$

where

$$\tan \theta = \frac{\operatorname{ctn} \delta_1 - \operatorname{ctn} \delta_2}{2}.$$

279. In order to obtain the temperature coefficient of a certain magnet, of moment M_1, it is placed in a water bath at a short distance from a magnetometer needle, its axis being perpendicular to the magnetic meridian at the centre of the needle. The needle is brought back to its zero position by a compensating magnet placed on the opposite side of the magnetometer at a distance d_0 from it, its axis being also perpendicular to the meridian at the centre of the needle. The moment of the compensating magnet is M_0, its magnetic

length $2\,l_0$. When the magnet M_1 is heated a given number of degrees, its moment decreases to M_1', and the magnetometer needle is deflected over n divisions of the scale. The scale distance being a, prove that

$$\frac{M_1 - M_1'}{M_1} = \frac{H}{M_0} \cdot \frac{(d^2{}_0 - l^2{}_0)^2}{4\,ad_0} \cdot n,$$

where the deflection n is small.

Show that if α_1 is the angle through which M_0 would deflect the needle if M_1 were absent,

$$\frac{M_1 - M_1'}{M_1} = \frac{\tan\alpha}{\tan\alpha_1}, \text{ where } \tan\alpha = \frac{n}{2\,a}.$$

280. Two magnets, m_1 and m_2, are placed, with their axes parallel to each other but opposite in direction, in Gauss's B position with respect to a magnetometer. The centre of m_1 is north of the magnetometer and the centre of m_2 south of it. The distances (d_1 and d_2) of the centres of m_1 and m_2 from the centre of the magnetometer needle are such that the needle is undeflected. Show that if μ_1 and μ_2 are the strengths of the "poles" of m_1 and m_2, and if $2\,l_1$, $2\,l_2$, and $2\,\lambda$ are the "lengths" of m_1, m_2, and the needle respectively, μ_1 is to μ_2 as

$$l_2\left\{\frac{1}{[l_2^2 + (d_2 - \lambda)^2]^{\frac{3}{2}}} + \frac{1}{[l_2^2 + (d_2 + \lambda)^2]^{\frac{3}{2}}}\right\}$$

is to
$$l_1\left\{\frac{1}{[l_1^2 + (d_1 - \lambda)^2]^{\frac{3}{2}}} + \frac{1}{[l_1^2 + (d_1 + \lambda)^2]^{\frac{3}{2}}}\right\}.$$

281. A fixed bar magnet of magnetic length $SN = 2\,L$ and of pole strength M, and a magnetic needle of magnetic length $sn = 2\,l$, of pole strength m, are in the same plane, with their centres (C, c) at a distance r from each other. The angles NCc and scC are equal to Φ and ϕ respectively. The lines

ns, *SN* meet when produced in *F*. The perpendicular distances of *N*, *C*, and *S* from *ns* are

$$r\sin\phi - L\sin(\phi+\Phi),\ r\sin\phi, \text{ and } r\sin\phi + L\sin(\phi+\Phi),$$

so that the length of the perpendicular dropped from *c* upon *Nn* is $l\sin(FnN)$ or $l[r\sin\phi - L\sin(\phi+\Phi)]/Nn$. The lengths of the perpendiculars dropped from *c* upon *Sn*, *Ns*, and *Ss* are

$$l\,[r\sin\phi + L\sin(\phi+\Phi)]/Sn,\ l\,[r\sin\phi - L\sin(\phi+\Phi)]/Ns,$$

and $$l\,[r\sin\phi + L\sin(\phi+\Phi)]/Ss.$$

Show that the sum of the moments, taken about *c*, of the forces which tend to decrease ϕ, is

$$D = Mm\left[\left\{\frac{l}{\overline{Nn}^3} + \frac{l}{\overline{Ns}^3}\right\}\left\{r\sin\phi - L\sin(\phi+\Phi)\right\}\right.$$
$$\left. - \left\{\frac{l}{\overline{Sn}^3} + \frac{l}{\overline{Ss}^3}\right\}\left\{r\sin\phi + L\sin(\phi+\Phi)\right\}\right]$$

or $$Mml\left\{r\sin\phi\left[\frac{1}{\overline{Nn}^3} + \frac{1}{\overline{Ns}^3} - \frac{1}{\overline{Sn}^3} - \frac{1}{\overline{Ss}^3}\right]\right.$$
$$\left. - L\sin(\phi+\Phi)\left[\frac{1}{\overline{Nn}^3} + \frac{1}{\overline{Ns}^3} + \frac{1}{\overline{Sn}^3} + \frac{1}{\overline{Ss}^3}\right]\right\}.$$

Show also that $\overline{Nn}^2 = r^2 + L^2 + l^2 + 2\,rl\cos\phi - 2\,rL\cos\Phi - 2\,l\,L\cos(\phi+\Phi)$, or $\dfrac{1}{\overline{Nn}^3} = \dfrac{1}{r^3}\left[1 + \dfrac{2\,l\cos\phi - 2\,L\cos\Phi}{r} + \dfrac{L^2 - 2\,l\,L\cos(\phi+\Phi) + l^2}{r^2}\right]^{-\frac{3}{2}}$, and that, if both *l* and *L* are small compared with *r* so that only the first powers of l/r and L/r need be kept, the approximate value $\dfrac{1}{r^3}\left[1 + \dfrac{3\,(L\cos\Phi - l\cos\Phi)}{r}\right]$ may be used for $\overline{Nn}^{-3}$. Treating $\overline{Ns}$, $\overline{Sn}$, and $\overline{Ss}$ in the same way, prove that if M_0 and

m_0 are the magnetic moments of the magnet and the needle respectively, we may use for D the approximate value

$$M_0 \, m_0 \left[3 \cos \Phi \sin \phi - \sin (\phi + \Phi)\right] / r^3.$$

A better approximation can be obtained by keeping higher powers of the ratios l / r and L / r. It is to be noticed that $\Phi = 0$ and $\Phi = 90^0$ correspond to Gauss's "Principal Positions."

282. Prove that the magnetic force at a large distance in the prolongation of its axis, due to a bar magnet of moment M, lies between $2 \, M / r_1^3$ and $2 \, M / r_2^3$, where r_1, r_2 are the distances from the two ends of the magnet.

283. If l, m, n are the direction cosines of the axis of a small magnetic needle free to turn about its centre in a magnetic field, and if L, M, N are the components of the couple which acts on the needle, $D_l L + D_m M + D_n N = 0$.

284. The accurately flat north end of one of two exactly similar, uniformly polarized, perfectly hard bar magnets is placed in close contact with the south end of the other, so that the two form a long, uniformly polarized, straight bar. What force is necessary to separate the magnets lengthwise? Compute the work necessary to separate into short elements a long, uniformly polarized, magnetic filament.

285. Has a polarized rigid distribution an axis in the sense that a straight bar magnet has a magnetic axis? Consider first a bent, solenoidally polarized, magnetic filament.

286. Show that if a polarized electrical distribution were enclosed in a thin "metallic skin connected with the earth," there would be induced upon the inner surface of the skin a charge, E, of total amount zero. Show also that the effect of the given distribution together with the charge on the inner surface of the skin would be nothing at outside points, and the effect at outside points of the given distribution the same as that of a charge on the skin equal to the negative of E. This charge is sometimes called "Green's Distribution" and sometimes "Poisson's Surface Distribution."

287. A solid soft-iron sphere is placed in a uniform magnetic field. Show that about three times as many lines of force pass through any closed curve within the sphere as through an equal and parallel curve at an infinite distance.

288. In the case of a certain sphere of radius a polarized parallel to the axis of x, $I \equiv A_0 \cdot r^{n-3}$, where r is the distance from the centre, and the function, f, mentioned on page 192, is r^{n-3}. Show that the values of the potential function within and without the sphere are

$$4\,\pi A_0 x r^{n-3}/n \text{ and } 4\,\pi A_0 a^n x/nr^3 \text{ respectively.}$$

Show that at the surface of the sphere the normal component of the induction is continuous, and the tangential components in general discontinuous. The tangential components of the force are continuous, and the normal component in general discontinuous, by the amount $4\,\pi\sigma$.

289. An uncharged conducting sphere of radius a is in a uniform field of force F, and consists of two hemispheres in contact with the plane of division perpendicular to the field. Show that if the hemispheres are separated, each will have a charge $3\,a^2F/4\,\pi$.

290. The field inside a shell bounded by two concentric spherical surfaces of radii a and b and uniformly polarized in the direction of the x axis, has the potential function zero. Outside the shell the potential function is $4\,\pi x I(b^3 - a^3)/3\,r^3$.

291. Show that $-3\,Xx/(\mu+2) + C$, for values of r less than a, and $-Xx + a^3Xx(\mu-1)/[r^3(\mu+2)] + C$, for values of r greater than a, represent the potential function within and without a sphere, of radius a, with centre at the origin, composed of a homogeneous dielectric of inductivity μ, placed in a uniform field in air of intensity X.

292. If a cylindrical surface which circumscribes an oval body P touches it in a curve which is the perimeter of a right section of the cylinder of area Q, and if P be uniformly polarized in the direction of the axis of the cylinder to

intensity I, the amount of matter in the positive distribution on P's surface is $Q \cdot I$. If an ellipsoid the semiaxes of which are a, b, c, be uniformly polarized to intensity I in the direction of the axis a, the moment, M, of the distribution is $\frac{4}{3}\pi abc I$ and the amount of matter, m, on either the positive or negative half is $\pi bc I$. The distance of the centre of gravity of either the positive or negative part of the distribution from the centre of the ellipsoid is $M/2m$ or $\frac{2}{3}a$. In what sense is the "magnetic length" of a uniformly polarized sphere $\frac{4}{3}a$?

293. In the case of any purely polarized distribution bounded by a surface S, the volume and superficial densities are accounted for by a vector I, of components A, B, C, such that within S, $\rho = -$ Divergence I, and on S, $\sigma = I \cdot \cos(n, I)$. Show that the polarization might be equally well represented by any vector which differs from I by a solenoidal vector Ω every line of which, if it meets S at all, lies wholly on S. Is the induction within a hard magnet definite?

294. Matter is distributed on the ends of a cylinder of revolution of length l and radius a. The density within the cylinder and the superficial density on its curved surface are everywhere equal to zero. On one end a quantity $2\pi a^3$ of matter is distributed with density $\sigma = a^2/r$, where r is the distance from the axis; on the other end a quantity $-2\pi a^3$ is distributed with density $\sigma = -3r$. Can you affirm that the cylinder is not polarized solenoidally?

295. Show that if, in the case of a polarization symmetrical about the axis of z so that the lines of the vector I lie in planes which pass through this axis, Z be the component of I parallel to the axis of z and R the component perpendicular to the axis, $\rho = -[D_r R + R/r + D_z Z]$. Consider the volume density in, and the superficial density on, a cylinder of revolution of length l and radius a, the axis of which coincides with the axis of z, when $R = (r-a)f(z)$, and

$$Z = \psi(r) - \int \{(r-a)f''(z) + (2r-a)f(z)/r\}\,dz.$$

Assuming both f and ψ at pleasure, draw the lines of polarization for the simple case which you have chosen.

296. A solenoidal vector, the components of which parallel to the columnar coördinates r, θ, x are $(a - r)f'(x)$, 0, $(2r - a)f(x)/r$, represents the polarization within a magnet bounded by the cylindrical surface $r = a$ and the planes $x = 0$, $x = b$. Determine the surface density σ and draw two of the lines of polarization when $f(x) \equiv x$. Show that σ is zero when $f(x) \equiv \sin(\pi x/b)$.

297. Show that a vector the components of which in the directions of the columnar coördinates r, θ, x are

$$[f'(x) \cdot F(r)],\ 0,\ -f(x)\,[F'(r) + F(r)/r],$$

is solenoidal, and use this form to determine two or three different polarizations within a bar magnet for which both ρ and σ shall be everywhere zero.

298. Prove that an infinitely long cylinder of revolution of radius a, the axis of which coincides with the z axis, when polarized uniformly in the direction of the x axis, gives rise to the potential function $-2\pi Ia^2x/r^2$ at outside points. This is identical with the potential function due to a plane doublet of strength $2\pi Ia^2$ at the origin. Within the cylinder the resultant force has the intensity $2\pi I$ and the direction of the negative x axis, while the induction has the intensity $2\pi I$ and the direction of the positive x axis. The lines of induction and the lines of force have the same direction without the cylinder and opposite directions within. The lines of force are shown in Fig. 133. Show that the normal component of the induction is continuous at the surface of the cylinder.

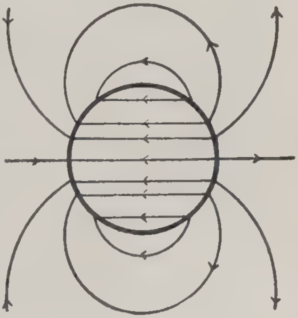

Fig. 133.

299. A solenoidally polarized distribution inside which the lines of polarization are straight and parallel need not be uniformly polarized.

300. Prove that the mutual potential energy of any two small magnets at a distance apart large compared with their linear dimensions is

$$M_1 \cdot M_2 (\cos\phi - 3 \cos\theta_1 \cdot \cos\theta_2) / r^3,$$

where M_1, M_2 are the moments of the magnets, ϕ the angle between their directions, and θ_1, θ_2 the angles which these directions make with a line drawn from the centre of the first to the centre of the second.

301. Show that for a simple magnetic shell in the form of a circle, the direction of the vector potential at any point is perpendicular to a plane through the point and a normal to the plane of the shell through the centre. [St. Peter's College.]

302. Prove that if m is the pole strength of a slender, straight, uniformly magnetized magnet AB, a vector potential may be found which has at any point P the value $\frac{m}{p}(\cos PAB + \cos PBA)$, where p is the length of the perpendicular dropped from P on AB, produced if necessary. Show that the direction of this vector potential is perpendicular to the plane PAB. [M. T.]

303. Show that if V is the value of the potential function, and F that of the vertical component of the magnetic force at the earth's surface, the earth's field in outside space may be considered as due to a surface distribution of density $-F/2\pi - V/4\pi a$, where a is the earth's radius.

304. A magnetic needle is placed near an infinite plane face of a mass of soft iron. Show that the reaction of the iron on the needle may be represented as due to a negative image of the needle in the plane face, reduced in intensity in the ratio of $(\mu - 1)/(\mu + 1)$, where μ is the permeability of the iron. [St. John's College.]

305. A magnetic element SN of pole strength a and moment b lies in a magnetic field which has the potential function V. Show that if l, m, n are the direction cosines of the axis of the element, the mutual potential energy of the element and the field is $\Delta E \equiv a(V_N - V_S) = b(l \cdot D_x V + m \cdot D_y V + n \cdot D_z V)$. If the element is a rectangular parallelopiped, $dx\,dy\,dz$, taken from a magnetized body in which the polarization is I, $b = I\,dx\,dy\,dz$, $\Delta E = (A \cdot D_x V + B \cdot D_y V + C \cdot D_z V)\,dx\,dy\,dz$, and the mutual energy of the field and the magnet is the integral of this last expression. If the magnet is a simple shell of strength Φ,

$$E = \Phi \int\int (l \cdot D_x V + m \cdot D_y V + n \cdot D_z V)\,dS, \text{ or}$$

$$-\Phi \int\int [X \cdot \cos(x, n) + Y \cdot \cos(y, n) + Z \cdot \cos(z, n)]\,dS,$$

where the integration is to be extended over one face of the shell.

306. A simple plane circular magnetic shell of radius r lies in the yz plane, with its centre at the origin, in a magnetic field symmetrical about the x axis. The intensity of the x component of the field is $F(x)$, where F is a continuous function, such that $F(\infty) = 0$. Show that the force which urges the shell is equal to $\pi a^2 \Phi \cdot D_x F$. The centre of the rigid shell is to move along the x axis to infinity while the plane of the shell is parallel to the yz plane. Compute the work done on the shell by the field during the motion. Has the field any component perpendicular to the x axis? Compute the work done on the shell by the field, with the help of the method discussed at the top of page 218. Show that a vector which has the components $F(x)$, $yC/(y^2+z^2) - \frac{1}{2}y \cdot F'(x)$, $zC/(y^2+z^2) - \frac{1}{2}z \cdot F'(x)$, is solenoidal and is symmetrical about the x axis.

307. Show that if A', B', C' are the components of magnetization at the point (x', y', z') in any magnet, M', and if p denotes the reciprocal of the distance between (x', y', z') and

(x, y, z), the components at (x, y, z) of the ordinary vector potential of the magnetic induction are

$$\iiint (B' \cdot D_{z'} p - C' \cdot D_{y'} p)\, d\tau',$$
$$\iiint (C' \cdot D_{x'} p - A' \cdot D_{z'} p)\, d\tau',$$
$$\iiint (A' \cdot D_{y'} p - B' \cdot D_{x'} p)\, d\tau'.$$

The scalar potential function of magnetic force is in the same notation

$$\iiint (A' \cdot D_{x'} p + B' \cdot D_{y'} p + C' \cdot D_{z'} p)\, d\tau'.$$

If M' is a simple shell of strength Φ', the x component of the vector potential function can be written in the form

$$\Phi' \iint [D_{z'} p \cdot \cos(y, n) - D_{y'} p \cdot \cos(z, n)]\, dS'. \quad \text{[Maxwell.]}$$

308. Show that if r is the distance from a fixed point, the line integral around any closed curve s of the tangential component of the vector $(1/r, 0, 0)$ is equal to the surface integral, taken over any cap S bounded by s, of

$$D_z(1/r) \cdot \cos(y, n) - D_y(1/r) \cdot \cos(z, n),$$

where n is a positive normal to the cap. Obtain two similar equations with the help of the vectors $(0, 1/r, 0)$, $(0, 0, 1/r)$, and prove that the components of the vector potential function of the force due to a magnetic shell of strength Φ in air are

$$\Phi \int [\cos(x, s) \cdot /r]\, ds, \qquad \Phi \int [\cos(y, s) \cdot /r]\, ds,$$
$$\Phi \int [\cos(z, s) \cdot /r]\, ds,$$

taken around the perimeter of the shell.

If (L, M, N) are the curl components of a vector (F_x, F_y, F_z), the latter is a vector potential function of the magnetic field.

Show that the mutual potential energy of a magnetic shell of strength Φ' and the field is $-\Phi'\int(F_x\cdot dx' + F_y\cdot dy' + F_z\cdot dz')$, taken around the shell in positive direction. If the external field is caused by another shell of strength Φ, we have

$$F_x \equiv \Phi\int[\cos(x,s)\cdot/r]ds,\quad F_y \equiv \Phi\int[\cos(y,s)\cdot/r]ds,$$
$$F_z \equiv \Phi\int[\cos(z,s)\cdot/r]ds,$$

where the integrals are to be taken around the perimeter s of the second shell, and the mutual potential energy of the two shells is

$$-\Phi\Phi'\int\int[\cos(x,s)\cdot\cos(x,s') + \cos(y,s)\cdot\cos(y,s')$$
$$+\cos(z,s)\cdot\cos(z,s')][ds\cdot ds'/r]$$

or

$$-\Phi\Phi'\int\int[\cos(ds,ds')/r]\,ds\cdot ds'.$$

The integral by which $-\Phi\Phi'$ is multiplied has been called the "geometric potential" of the two curves.

309. Prove in two different ways that the energy of the surface distribution $\sigma = I\cdot\cos(n,I)$, on a sphere of radius a uniformly polarized to intensity I, is $8\pi^2I^2a^3/9$. In what sense is this the energy of the distribution? Give a summary of the reasoning of Lord Kelvin in his paper "On the Mechanical Values of Magnets."

310. If a polarized distribution is placed in a field of force which has a potential function V, the mutual potential energy of the field and the distribution as a whole is

$$\int\int VI\cdot\cos(n,I)\,dS - \int\int\int V(D_xA + D_yB + D_zC)\,d\tau,$$

where the first integral is to be extended over the surface of the distribution and the second through its volume. Show that this energy is equivalent to

$$\int\int\int(A\cdot D_xV + B\cdot D_yV + C\cdot D_zV)\,d\tau.$$

311. A sphere of radius a uniformly polarized to intensity I is placed in a uniform field of force of intensity X. Show that if the directions of the field and the polarization coincide, the mutual potential energy of the sphere and the field is $-4\pi a^3IX/3$. What would be the energy if the direction of the field and polarization were opposed? It would be zero if these directions were perpendicular to each other.

312. If V is the potential function due to a volume distribution of density ρ_1 in a region T_1, and a surface distribution of density σ_1 on a surface S_1, and if U is a continuous function

$$\begin{aligned}4\pi\int\!\!\int\!\!\int U\rho_1\cdot d\tau_1 + 4\pi\int\!\!\int U\sigma_1\cdot dS_1 \\ =\int\!\!\int\!\!\int(D_xU\cdot D_xV + D_yU\cdot D_yV + D_zU\cdot D_zV)d\tau \\ -\int\!\!\int U\cdot D_rV\cdot dS,\end{aligned}$$

where the volume and surface integrations in the second member are to be extended respectively through and over a spherical surface of radius r so large as to include T_1 and S_1. If U vanishes at infinity, the last surface integral vanishes when r is infinite. Use this equation to compute the mutual potential energy $(-\frac{4}{3}\pi a^3IX)$ of a sphere of radius a, uniformly polarized to intensity I in the direction of the x axis, and a uniform field $(X, 0, 0)$, in which it lies. In this case the value of the last term in the second member is $-32\pi^2a^3IX/9$.

313. At a distance of 10 centimetres from the middle point of a wire 140 centimetres long, the magnetic force due to a current in the wire would be within one per cent of that which would be produced if the wire were infinite.

314. Show that the magnetic force within a square circuit (of side $= 2a$) at a point midway between two sides, at a distance x from the centre of the square, is

$$\frac{2C}{a}\left\{\frac{\sqrt{a^2+(a-x)^2}}{a-x}+\frac{\sqrt{a^2+(a+x)^2}}{a+x}\right\}.$$

Draw a curve which shall represent this force as a function of x. What if x is greater than a? Show that the force at a point distant y from the plane of the circuit in the axis of the circuit is

$$\left\{\frac{8\,Ca^2}{y^2+a^2}\cdot\frac{1}{\sqrt{2\,a^2+y^2}}\right\}.$$

Show that if a current of A amperes be sent through a tangent galvanometer which has a square coil consisting of n turns of wire,

$$A=\frac{5\,aH\tan\delta}{2\sqrt{2}\cdot n}.$$

315. If a circuit carrying a steady current C is a regular polygon of $2\,n$ sides, and if a is the radius of the inscribed circle, the magnetic force at the centre is $(4\,n\,C/a)\cdot\sin(\pi/2\,n)$.

316. The plane of the ring of a tangent galvanometer which consists of a single turn of fine wire is the vertical plane of the magnetic meridian. Show that if a current of A amperes be sent through the ring, the strength of the field at a point P in its axis at a distance z from the centre is $\dfrac{\pi r^2 A}{5\,(z^2+r^2)^{\frac{3}{2}}}$, where r is the radius of the ring. Hence prove that if the centre of the galvanometer needle is at P, the deflection will be given by the equation $A=[\,5\,(z^2+r^2)^{\frac{3}{2}}\,H\tan\alpha]\,/\,\pi r^2$.

317. Show that at a point on the axis, at a short distance (z) from the centre of a tangent galvanometer coil of radius a, the intensity of the electromagnetic field due to a steady current passing through the coil is to the intensity of the same field at the centre as $\left(1-\dfrac{3\,z^2}{2\,a^2}\right)$ to 1, nearly.

318. Show that if around a ring formed of a piece $2\,b$ centimetres long of a thin metal tube of inside radius a and of outside radius $a+\delta$, a steady current of strength $2\,b\delta C$ uniformly distributed through the conductor could be sent, the

strength of the resulting electromagnetic field at the centre of the axis of the coil would be $\frac{4\pi C b\delta}{\sqrt{b^2+a^2}}$. What would this intensity become if the tube were to shrink indefinitely in length while the whole current around it remained unchanged? Assuming that when x is small

$$(1+x)^{-\frac{1}{2}} = 1 - \tfrac{1}{2}x + \tfrac{3}{8}x^2 - \tfrac{5}{16}x^3 + \cdots,$$

deduce from your results the usual correction, $-\frac{b^2}{2a^2}$, for the breadth of the coil of a tangent galvanometer.

319. The vertical coil of a tangent galvanometer makes a small angle δ with the east and west line through the centre of its needle. If a steady current, of such strength that it would cause a deflection of 45° if the plane of the coil were in the meridian, be now sent through the coil, it will cause a deflection of $\frac{1}{2}\delta$.

320. The centres of the rings of a two-coil tangent galvanometer are 20 centimetres apart and the mean radius of each of the coils is 20 centimetres. The centre of the needle is on the common axis of the coils halfway between their centres. When the instrument is properly set up in a certain place a steady current of half an ampere sent through both coils in series causes a deflection of 45°. Show that if there are 20 turns in each coil, $H = 16\pi/10\,(5)^{\frac{3}{2}}$.

321. A tangent galvanometer has two equal vertical coils, each of mean radius r, placed at a distance apart of $2r(\sqrt[3]{4}-1)^{\frac{1}{2}}$. The short compass needle is placed midway between the coils on their common axis. Show that the needle deflection caused by any current which passes in the same direction through both coils in series will be the same as if the same current passed through only one coil, while the centre of the needle was at the centre of this coil.

322. Show that if the vertical coil of a tangent galvanometer makes an angle θ with the meridian, and if a current

of C amperes be sent through it first in one direction and then in the other, causing deflections of δ_1 and δ_2 respectively, then

$$\frac{\pi n C}{5 r H} = \frac{\sin \delta_1}{\cos(\theta - \delta_1)} = \frac{\sin \delta_2}{\cos(\theta + \delta_2)},$$

and $\tan\theta = \frac{1}{2}(\operatorname{ctn}\delta_2 - \operatorname{ctn}\delta_1)$, where r is the mean radius of the coil and n the number of turns of wire on it.

323. From a thin, flat sheet of copper of thickness δ is cut a ring of inside radius $a - d$ and outside radius $a - d$. If a steady current of strength $2C\delta d$ could be made to circulate around this ring, what would be the strength of the electromagnetic field at the centre of the ring? What would this strength become if the ring were to shrink to a fine wire ring of radius a concentric with the original ring without change of the current strength? Assuming that when x is small

$$\log\frac{1+x}{1-x} = 2[x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \cdots],$$

deduce from your results the usual correction (one-twelfth of the square of the ratio of the depth of the coil to its mean radius) for the depth of the ring of a tangent galvanometer.

324. A certain galvanometer coil is wound upon a large square frame. When the vertical plane of the coil makes an angle θ with the meridian a certain current C sent through the coil deflects the short needle through an angle $\frac{1}{2}\theta$ towards the coil. Show that C would cause the same deflection if the coil were in the meridian.

325. On the axis of a fixed circular ring of wire which carries a steady current C is a molecular magnet of moment m. Show that if the axis of the magnet makes an angle θ with the axis of the ring the moment of the couple which tends to diminish θ is $(2\pi m C \sin^3\phi \cdot \sin\theta)/a$, where a is the length of a radius of the ring and ϕ the angle subtended at the molecule by the radius.

326. A perfectly flexible wire fastened at two fixed points carries a current of given strength. Prove that in a uniform field of magnetic force it will tend to assume the shape of a helix. [M. T.]

327. A very long straight wire which carries a steady current C is at right angles to the plane of a circular ring of radius a which carries a current C'. The ring is free to turn about the diameter which intersects the straight wire. Prove that the couple tending to turn the ring is $2\pi CC'a^2/r$ or $2\pi CC'r$, according as a is less or greater than r, the distance of the wire from the centre of the ring. [Trinity College.]

328. A plane ring can move about a diameter parallel to an infinite straight wire, the distance of which from the centre of the ring is equal to the radius of the latter. Show that when currents CC' are sent through the two circuits the couple tending to turn the ring is

$$4\pi CC'a(\cos\phi - \cos^{\frac{3}{2}}\phi/\sqrt{2\cos\phi}),$$

when a is the radius of the ring and ϕ the acute angle which the normal to its plane makes with the perpendicular to the straight wire drawn from the centre. [M. T.]

329. If a layer of n' turns of wire carrying a steady current of unit strength and forming a coil k' be wound uniformly on such a ring coil, k, as that shown in Fig. 77, the induction due to the current in k' has at every point within the coil the value $2\mu n'/r$. The integral of n times this quantity taken over a cross-section of the ring R on which k is wound gives the mutual inductance of the two coils. Show that if R may be regarded as formed by revolving a circle of radius a about a line in its plane, distant b from its centre, the value of the integral is $4\pi\mu nn'(b - \sqrt{b^2 - a^2})$. If R were formed by the revolution of a rectangle with sides of length b parallel to the axis and a perpendicular to it, the value of the integral would be

$$2nn'\mu b\log\frac{c + a/2}{c - a/2},$$

where c is the distance of the centre of the section of the ring from the axis. Show that the self-inductance of k might be found for the two cases just mentioned by putting n equal to n' in the expressions for mutual inductance, and imagining k' to move into coincidence with k.

330. A thin tubular conductor of circular section has a radius a and carries a steady current C; prove that the mechanical action between the different portions of the current produces a transverse tension in the tube, of intensity $C^2/\pi a$. [St. John's College.]

331. The ponderomotive forces which act upon a portion A_1A_2 of a circuit which carries a steady current of strength C in the field of a magnetic pole of strength m at the point O, have a resultant moment M about any straight line OZ drawn through O. Let PP' represent an element Δs of the circuit; let $OP = r$, $OP' = r + \Delta r$, $ZOP = \theta$, $ZOP' = \theta + \Delta\theta$, $(r, s) = \delta$, and denote the angle between the planes ZOP and POP' by ϕ. The fundamental equation of spherical trigonometry yields

$$\cos(\theta + \Delta\theta) = \cos\theta \cdot \cos POP' + \sin\theta \cdot \sin POP' \cdot \cos\phi,$$

and it is evident, since $\Delta\theta$ is not greater than POP', that the limit of the ratio of $(\cos\Delta\theta - \cos POP')/\sin POP'$ is zero, so that $\cos\phi$ is approximately equal to $-\sin\Delta\theta/\sin POP'$. The Theorem of Sines applied to the plane triangle POP' yields the equation $PP'/OP' = \sin POP'/\sin OPP'$. Prove that the moment about OZ of the elementary force exerted by the pole upon Δs may be written,

$$\Delta M = mC \cdot \sin\delta \cdot \cos\phi \cdot \sin\theta \cdot \Delta s/r,$$

and that for purposes of integration this is equivalent to $-mC \cdot \sin\theta \cdot d\theta$, so that $M = mC(\cos\theta_2 - \cos\theta_1)$, where $\theta_1 = ZOA_1$, $\theta_2 = ZOA_2$. If $\theta_1 = \theta_2$, as in the case of a closed circuit, M is zero. Consider the possibility of rotation about a straight line, of a closed circuit bearing a steady current C under the action of any number of magnetic poles on the line.

332. In the case of a solitary linear circuit s carrying a current C in its own magnetic field, all the lines of force are closed curves threading the current, and the line integral of the force taken around any one of these curves is $4\pi C$. The equipotential surfaces fill all space; each of them is a cap bounded by the circuit, and the surface integrals of the induction taken over these caps are all equal. Use the reasoning of page 270 to show that since

$$4\pi C = \int H \cdot ds, \text{ and } p = \int\int B \cdot dS,$$

$$4\pi Cp = \int\int_\infty\int H \cdot B \cdot d\tau,\ 4\pi C(p + dp) = \int\int_\infty\int H(B + dB)\, d\tau$$

$$4\pi C \cdot dp = \int\int_\infty\int H \cdot dB\, d\tau,$$

where dp is the increment of the induction flux through the circuit, due to a small increase in the current. Show from the equation $E - dp/dt = rC$ that, besides the energy dissipated in heat, the generator in a solitary circuit must furnish an amount of energy $C \cdot dp$ while the current in the circuit is changed from C to $C + dC$, and that the difference dW between this quantity and the increment dT of the electrokinetic energy shows the amount of energy which is used in some other way than in increasing this energy.

Prove that

$$dW = \frac{1}{8\pi}\int\int_\infty\int [2\, H \cdot dB - (H \cdot dB + B \cdot dH)]\, d\tau$$

and use this expression to compute (see page 291) the energy loss due to hysteresis during a cycle of magnetization.

333. The distance between the axes of two infinitely long, straight, round, non-magnetic wires (A_1, A_2) of radius a and parallel to each other is b. One wire carries a steady current C, uniformly distributed, in one direction, and the other wire an equal current, uniformly distributed, in the other direction.

If the cross-section of each wire be divided into n elements of equal area, every element, dS, is the section of a filament which carries a current $CdS/\pi a^2$. Imagine a circuit made up of a certain filament F_1 in A_1, distant r_1', r_1'' from the axes of the wires, and a filament F_2 in A_2 distant r_2', r_2'' from these axes. The flow of induction through this circuit due to A_1 is

$$\int_{r_1'}^{a} \frac{2\,Cr}{a^2}\,dr + \int_{a}^{r_2'} \frac{2\,C}{r}\,dr \text{ or } 2\,C[(a^2 - r_1'^2)/2\,a^2 + \log(r_2'/a)]$$

and that due to A_2 is $2\,C[(a^2 - r_2''^2)/2\,a^2 + \log(r_1''/a)]$. If the filaments are symmetrically situated, $r_2' = r_1''$, $r_1' = r_2''$ and the induction through the circuit is

$$4\,C[(a^2 - r_1'^2)/2\,a^2 + \log(r_1''/a)].$$

The electrokinetic energy of a set of circuits is equal to one-half the sum of the products formed by multiplying the induction through any circuit by the current in that circuit. The contribution which the elementary circuit just mentioned would make to the electrokinetic energy T is, therefore, one-half the product of the induction through it and the current which it carries, so that

$$T = \frac{C^2}{\pi a^2}\int\int \{1 - r_1'^2/2\,a^2 - r_2''^2/2\,a^2 + \log(r_2'/a) \\ + \log(r_1''/a)\}\,dS,$$

where the integration is to be extended over all such elementary circuits, that is, over the cross-section of either wire. We may write for dS, either $r_1' \cdot dr_1' \cdot d\theta_1$ or $r_2'' \cdot dr_2'' \cdot d\theta_2$, at pleasure, and we may use the first of these for the first, second, and fifth terms of the integrand, and the second form for the other terms. The limits of θ will be 0 and $2\,\pi$, and those of r_1' and r_1'', 0 and a. Assuming that, if $m > n$,

$$\int_0^{\pi} \log(m + n\cos\theta)\,d\theta = \pi \log\{\tfrac{1}{2}(m + \sqrt{m^2 - n^2})\},$$

show that $T = C^2[\frac{1}{2} + \log(b^2/a^2)]$ and that it makes no difference how the separate filaments of A_1 and A_2 are combined into elementary circuits. Show also that the inductance, per unit of its length, of the circuit made up of the wires, is $2\log(b^2/a^2) + 1$. Prove that if the wires had inductivities μ_1 and μ_2 and radii a_1, a_2, we should have

$$L = 2\,\mu \log/(b^2 a_1 a_2) + \tfrac{1}{2}(\mu_1 + \mu_2),$$

where μ is the inductivity of the surrounding medium. [For a discussion of the inductance of the circuit when the cross-sections of the long parallel wires are of any form, the reader is referred to A. Gray's *Absolute Measurements in Electricity and Magnetism*, Vol. II, p. 288, and to Drude's *Physik des Aethers*, p. 207.]

334. Obtain Heaviside's expressions (*Electrical Papers*, p. 101) for the coefficients L_1, L_2, M, of self and mutual induction for two parallel wires of length l, radii a_1, a_2, and inductivities μ_1, μ_2, suspended parallel to each other and to the earth at heights h_1, h_2 and at a horizontal distance d apart, if the current is supposed to return through the earth in a thin sheet, and if h_1 and h_2 are small compared with l. These expressions are

$$L_1/l = \tfrac{1}{2}\mu_1 + 2\log(2\,h_1/a_1),\ L_2/l = \tfrac{1}{2}\mu_2 + 2\log(2\,h_2/a_2),$$

$$M/l = \log\frac{d^2 + (h_1 + h_2)^2}{d^2 + (h_1 - h_2)^2}.$$

335. Show that if K and μ represent the dimensions of electric and magnetic inductivities respectively, the dimensions of μ in terms of L, M, T, K are $L^{-2}T^2K^{-1}$, while those of K in terms of L, M, T, μ are $L^{-2}T^2\mu^{-1}$. Show that the dimensions of electric quantity in the two systems are $L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}K^{\frac{1}{2}}$, $L^{\frac{1}{2}}M^{\frac{1}{2}}\mu^{-\frac{1}{2}}$; those of magnetic quantity $L^{\frac{1}{2}}M^{\frac{1}{2}}K^{-\frac{1}{2}}$, $L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}$; those of electric field strength $L^{-\frac{1}{2}}M^{\frac{1}{2}}T^{-1}K^{-\frac{1}{2}}$, $L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-2}\mu^{\frac{1}{2}}$; those of magnetic field strength $L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-2}K^{\frac{1}{2}}$,

$L^{-\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}$; those of electric potential $L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}K^{-\frac{1}{2}}$, $L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}\mu^{\frac{1}{2}}$; those of magnetic potential $L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}K^{\frac{1}{2}}$, $L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}$; those of conductivity $LT^{-1}K$, $L^{-1}T\mu^{-1}$; those of electric current $L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}K^{\frac{1}{2}}$, $L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}$; those of capacity LK, $L^{-1}T^{2}\mu^{-1}$; those of inductance $L^{-1}T^{2}K^{-1}$, $L\mu$; those of magnetic moment $L^{\frac{3}{2}}M^{\frac{1}{2}}K^{-\frac{1}{2}}$, $L^{\frac{5}{2}}M^{\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}$; those of electric surface density $L^{-\frac{1}{2}}M^{\frac{1}{2}}T^{-1}K^{\frac{1}{2}}$, $L^{-\frac{3}{2}}M^{\frac{1}{2}}\mu^{-\frac{1}{2}}$.

336. A rigid plane wire of any shape is free to turn about a point O in its plane distant a and b from the nearer and farther ends of the wire. The plane of rotation is perpendicular to the lines of a uniform field of induction of intensity B. Show that if the wire forms part of a circuit which carries a current C, the moment about O of the forces which, acting on the wire tend to set it in motion, is $\frac{1}{2}BC(b^2 - a^2)$. If the wire rotates with angular velocity ω, it cuts the lines of the field at the rate $\frac{1}{2}\omega B(b^2 - a^2)$.

337. A copper disc perpendicular to the lines of a uniform magnetic field is spun in its own plane about a fixed point O and is continuously touched at two points by the fixed electrodes of a galvanometer. Show that the current in the galvanometer is proportional to the difference of areas swept out by the radii vectores from O to the points touched. [M. T.]

338. A magneto-electric machine, driven at a constant rate, sends current through the coil of another magneto-electric machine used as a motor. When the second machine is held still, a power W is used in the circuit. Prove that the maximum power obtainable from the second machine is $\frac{1}{4}W$, and that then the first machine absorbs $\frac{1}{2}W$ from the engine which drives it. [M. T.]

339. Show that (1) if a conductor be moved along a line of magnetic induction parallel to itself, it will experience no electromotive force; (2) if a conductor carrying a current be free to move along a line of magnetic induction, it will experience no tendency to do so; (3) if a linear conductor coincide

in direction with a line of magnetic induction and be moved parallel to itself in any direction, it will experience no electromotive force in the direction of its length; (4) if a linear conductor carrying an electric current coincide in direction with a line of magnetic induction, it will not experience any mechanical force. [Maxwell.]

340. Discuss the following statements of different writers: "When an electromotive force E is suddenly applied to an inductive circuit of resistance R, the counter-electromotive force of self-induction is initially equal to E and the current caused by E is initially zero." "When the current is rising a portion of E, viz., RC, is employed in maintaining according to Ohm's Law the current C already established; the other portion of E, viz., $L \cdot D_t C$, is employed in increasing the electromagnetic momentum LC." "At the beginning the whole of the electromotive force acts to increase the current." "If a current is established in a coil and the coil left to itself, short circuited without any electromotive force to maintain the current, then as the decaying current reaches a value C the electromotive force RC is equal to $-L \cdot D_t C$."

"The reactance does not represent the expenditure of power, as does the effective resistance, r, but merely the surging to and fro of energy. While the effective resistance, r, refers to the energy component of the applied electromotive force or the electromotive force in phase with the current, the reactance, x, refers to the wattless component of the electromotive force or the electromotive force in quadrature with the current."

341. Compare the differential equation of motion of a body of mass L, moving with velocity C, under the action of an impressed force E, which tends to increase the velocity, and a resistance rC, proportional to the velocity, with the equation which the current in an inductive circuit must satisfy. Why should LC in the electrical case be called the "electromagnetic momentum"?

342. Given that

$$\int \sin pt \cdot \sin(pt + \alpha) \cdot dt$$
$$= [(2pt - \sin 2pt)\cos\alpha - \cos 2pt \cdot \sin\alpha]/4p,$$

show that the average value for any number of whole periods of the product of two simple harmonic functions of the same period is half the cosine of their phase difference. Show that the activity, or "power," of a harmonic alternating current is equal to the product of the effective current, the effective electromotive force applied to the circuit, and the cosine of their difference of phase.

343. Two coils, the resistances of which are r_1, r_2 and the inductances L_1, L_2, are in series in a simple circuit carrying a harmonic current. Is the impedance of the two taken together equal to the sum of their impedances?

344. The ends of the coil of an electromagnet are subject to a rapidly alternating electromotive force. Show that the energy expended in the battery when a given amount of heat is produced in the wire will be greater than would be the case if the electromotive force were constant in direction and magnitude.

345. Show that if a number of linear circuits $s_1, s_2, s_3, \cdots$, carrying currents $C_1, C_2, C_3, \cdots$, exist together, and if the total flow of induction through s_k be denoted by p_k, the electrokinetic energy, T, may be written $\frac{1}{2}\Sigma C_k p_k$ and $p_k = \frac{1}{2} D_{c_k} T$. The quantity p_k is sometimes called the *electrokinetic momentum of* s_k. Compare this result with the equation, $p_k = D_{c_k} T$, given on page 296.

346. Show that the total flux of electric current induced in a thin circular coil of radius a and resistance R, made up of n turns of wire, when the coil is turned through two right angles in the earth's uniform magnetic field H, is $2\pi a^2 n H/R$. Show also that if a sphere of soft iron of the same radius a be pushed completely into the opening in the coil, the flux is increased in the ratio of $3\mu/(\mu + 2)$. [St. John's College.]

347. A, B, C are three points in order in a simple circuit carrying an alternating current; between A and B is a non-inductive resistance R, and between B and C an unknown inductive resistance. The effective difference of potential, as measured by a voltmeter between A and B, is E_1, that between B and C is E_2, and that between A and C, E_3. Draw a diagram for this problem something like that shown in Fig. 107 and prove that the phase of the impressed electromotive force between A and C is in advance of the phase of the current by θ, where $\cos\theta = (E_1^2 + E_3^2 - E_2^2)/2\,E_1E_3$, and that the activity in the circuit between A and C is $(E_1^2 + E_3^2 - E_2^2)/2\,R$. What is the impedance between B and C? What is the "Three Voltmeter Method" of measuring the power in an alternating circuit?

348. A 100-volt lamp the resistance of which is 100 ohms is to use current from the mains of an alternating system of frequency 50, which have an effective potential difference of 200 volts. Show that it is better to use in series with the lamp a choking coil of negligible resistance but of such self-inductance $[\sqrt{3}/\pi]$ as shall reduce the current to the proper strength rather than a simple 100-ohm resistance coil. Show that while the saving is 100 watts, the lag of current, when the choking coil is used, is 60°.

349. The poles of a condenser of 5 microfarads capacity are joined by a resistance of 100 ohms the self-inductance of which is 0.5 henry. The current in the simple circuit is $25 \cdot \sin 500\,t$. Show that the maximum impressed electromotive force is about 4500 volts and the maximum potential difference between the plates of the condenser 10,000 volts.

350. A condenser of 2 microfarads capacity charged to 100 volts, is to be discharged through a circuit of resistance 10 ohms and self-inductance 0.05 henry. Show that the discharge will be oscillatory, with a period of about 0.002 second. After ten oscillations the amplitude will have about 0.135 its original value.

351. An alternating electromotive force of frequency 200 and effective intensity 200 volts, applied to a certain inductive circuit of 5 ohms resistance, causes an effective current of 10 amperes to flow through the circuit. Show that the current lags behind the electromotive force by about 75° 30′. What would be the effect of introducing a condenser of 41 microfarads capacity into the circuit?

[For examples of resonance the reader is referred to Professor Pupin's papers in the *American Journal of Science*, 1893.]

352. A condenser is discharged by means of a circuit which contains a hoop revolving in a magnetic field. Show that the discharge current satisfies an equation of the form

$$L \cdot D_t^2 C + R \cdot D_t C + C / K = M \cdot \sin pt.$$

Show also that the cases $KR^2 < 4L$, $KR^2 > 4L$, $KLp^2 = 1$ should be discussed separately. [St. John's College.]

353. An alternating electromotive force the maximum value of which is 1414 volts and the frequency $200/\pi$ is applied to the extremities of a circuit of resistance 300 ohms and of inductance 1 henry. Show that the reading of an ammeter in the circuit is 2.

354. An alternate current the frequency of which is $10{,}000/2\pi$ passes through a telephone of self-inductance 0.01 henry. The resistance of the circuit, which is otherwise non-inductive, is 10 ohms. Show that if the resistance of the circuit were increased seventeenfold, the current would be reduced about one-half. What would be the effect of introducing into the circuit a condenser of capacity 1 microfarad?

355. A simple harmonic electromotive force, the maximum value of which is 400 volts, is applied to an inductive circuit, the resistance of which is 2 ohms; the counter-electromotive force has an average value of 200 volts. Show that the power supplied to the circuit is about 26.8 horse power.

356. Show that the equations given on page 307 for the discharge of a condenser charged to potential Q_0/k may be written in the forms

$$C = -2\,Q_0 \cdot e^{-rt/2L} \cdot \sin mt \cdot / \sqrt{4\,kL - k^2r^2},$$

$$Q = 2\,Q_0 \cdot e^{-rt/2L} \cdot \sqrt{kL} \cdot \sin(mt + \alpha) \cdot / \sqrt{4\,kL - k^2r^2},$$

where $m^2 \equiv (4\,kL - k^2r^2)/4\,k^2L^2$,

and $\tan \alpha \equiv 2\,Lm/r$.

357. Two conducting circuits ACB and ADB in multiple arc (Fig. 108) carry the current $M \cos pt$ from A to B. Prove that if r and s are the resistances of the two branches and L, 0 their inductances, and if p is large, the heat generated per second in ACB is to the heat generated per second in ADB as rs to $(L^2p^2 + r^2)$.

358. $ABCD$ is a Wheatstone net. A coil of inductance L is inserted in AD, and the points A and D are connected to the terminals of a condenser of capacity K. Show that if $RK^2 = L$, where R is the resistance of the arm AD, no electricity will pass through a galvanometer in the arm BD when the circuit is closed. [M. T.]

359. Two similar alternators, each of internal resistance r_0 and self-inductance L, driven on a single shaft, are joined up in series with an exterior non-inductive resistance $2\,r$; the electromotive forces of the generators are $E_m \cdot \sin(pt + \theta)$, $E_m \sin(pt - \theta)$, and their sum is $2\,E_m \cos\theta \cdot \sin pt$. Show that the current in the circuit is

$$E_m \cdot \cos\theta \cdot \sin(pt - \delta) / \sqrt{(r + r_0)^2 + L^2p^2},$$

or $G \cdot \cos\theta \cdot \sin(pt - \delta)$, where $\tan\delta = Lp/(r + r_0)$. Of the whole power, $E_mG \cdot \cos^2\theta \cdot \cos\delta$, in the circuit, the first machine furnishes the quantity, $\frac{1}{2} E_mG \cdot \cos\theta \cdot \cos(\theta + \delta)$, and the second machine the greater quantity, $\frac{1}{2} E_mG \cdot \cos\theta \cdot \cos(\theta - \delta)$. Discuss the case where these quantities have opposite signs.

Show that if the machines were driven at practically the same speed by independent motors, the first would tend to increase its lead until the electromotive forces were opposed.

360. If two similar alternators, each of internal resistance r and inductance L, and driven at the same speed, be joined up in parallel to form a single generator used to send a current through an outside non-inductive resistance R, if C_1 and C_2 are the currents which pass through each of the machines, and if their electromotive forces are $e_1 \equiv e_0 \cdot \sin(pt + \alpha)$, $e_2 \equiv e_0 \sin(pt - \alpha)$; we have

$$e_1 - L \cdot D_t C_1 = C_1(r + R) + C_2 R,$$

and $$e_2 - L \cdot D_t C_2 = C_2(r + R) + C_1 R;$$

or $$(r + R + L \cdot D_t)\, C_1 + R C_2 = e_1,$$

and $$R C_1 + (r + R + L \cdot D_t)\, C_2 = e_2.$$

If we apply the operation $(r + R + L \cdot D_t)$ to the first of the last pair of equations and the operation R to the second, and subtract one resulting equation from the other, we shall get $[(r + R + L \cdot D_t)^2 - R^2] C_1 = (r + R + L \cdot D_t)\, e_1 - R e_2$; a similar equation for C_2 is easily found. Write down values for C_1 and C_2. Show that the leading alternator is doing more work than the other and that there is a general tendency towards synchronism. An interesting discussion of this problem may be found in Perry's *Calculus for Engineers.*

361. If a magnetic field which changes with the time has the components $X = m \cdot \sin pt$, $Y = n \cdot \sin pt$, $Z = 0$, the effect is the same as if a field of constant uniform intensity $\sqrt{m^2 + n^2}$ were rotating with constant angular velocity p.

362. If an alternating current C be divided at the point A between two conductors of ohmic resistance r_1, r_2, in which the current strengths are C_1 and C_2, $C^2 = C_1^2 + 2\, C_1 C_2 + C_2^2$ and Mean value of C^2 = Sum of mean values of C_1^2, C_2^2, and $2\, C_1 C_2$. Show that the current in the main circuit, as measured by the indication (I) of an ammeter in it, is not equal to the sum of

the currents (I_1, I_2), similarly measured, in the two branches. If one of the branches (r_1) of the divided circuit is non-inductive, the instantaneous difference of potential between its ends is $r_1 C_1$ and the instantaneous rate of expenditure of power in the other branch is $r_1 C_1 C_2$; the average activity in this latter branch is r_1 times the average value of $C_1 C_2$. Show that the power expended in r_2 is $\frac{1}{2} r_1 (I^2 - I_1^2 - I_2^2)$. What is the "Three Ammeter Method" of measuring power?

363. Show that if $u + vi$ is a solution of the equation

$$L \cdot D_t C + rC = E_0 \cdot e^{i(pt+a)},$$

v is a solution of the equation

$$L \cdot D_t C + rC = E_0 \cdot \sin(pt + a).$$

Prove that the complete solution of this last equation is

$$Ae^{-rt/L} + E_0 \cdot \sin(pt + b) / \sqrt{r^2 + L^2 p^2},$$

where $\tan b = (r \cdot \sin a - Lp \cdot \cos a) / (r \cdot \cos a + Lp \cdot \sin a)$.

364. Show that if u and v are solutions of the equations

$$L \cdot D_t C + rC = E_1 \cdot \sin(p_1 t + a_1),$$
$$L \cdot D\,C + rC = E_2 \cdot \sin(p_2 t + a_2)$$

respectively, $u + v$ is a solution of the equation

$$L \cdot D\,C + rC = E_1 \cdot \sin(p_1 t + a_1) + E_2 \cdot \sin(p_2 t + a_2),$$

and write down the complete solution of this last equation. Write down also an expression for the current in an inductive circuit to which n simple harmonic electromotive forces of given periods are applied.

365. Show that if $u + vi$ is a solution of the equation

$$r \cdot D_t C + C/k = pE_0 e^{i(pt+a)},$$

u is a solution of the equation

$$r \cdot D_t C + C/k = pE_0 \cdot \cos(pt + a).$$

Prove that the complete solution of this last equation is

$$Ae^{-t/kr} + kpE_0 \sin(pt + \beta) / \sqrt{1 + r^2p^2k^2},$$

where $$\tan\beta = (rpk \sin\alpha + \cos\alpha)/(rpk\cos\alpha - \sin\alpha).$$

366. Show that if u and v are solutions of the equations

$$r \cdot D_tC + C/k = A_1 \cdot \cos(p_1t + a_1),$$

$$r \cdot D_tC + C/k = A_2 \cdot \cos(p_2t + a_2),$$

$u + v$ will be a solution of the equation

$$r \cdot D_tC + C/k = A_1 \cdot \cos(p_1t + a_1) + A_2 \cdot \cos(p_2t + a_2),$$

and write down the complete solution of this last equation. Write down also an expression for the current in a non-inductive circuit of capacity k to which n simple harmonic electromotive forces of given periods are applied.

367. If ϕ stands for the operation D_t, ϕ^2 for the operation D_t^2, and so on, the result of applying the operation

$$(p_0 + p_1\phi + p_2\phi^2 + p_3\phi^3 + \cdots)$$

to e^{kt} is equal to the product of e^{kt} and

$$(p_0 + p_1k + p_2k^2 + p_3k^3 + \cdots).$$

Show that if we denote the result of applying the operation

$$Z \equiv (p_0 + p_1\phi + p_2\phi^2 + \cdots)/(q_0 + q_1\phi + q_2\phi^2 + \cdots)$$

to e^{kt} by u, so that u satisfies the equation

$$(q_0 + q_1\phi + q_2\phi^2 + \cdots)\,u = (p_0 + p_1\phi + p_2\phi^2 + \cdots)\,e^{kt},$$

a special value of u is the product of e^{kt} and the fraction

$$(p_0 + p_1k + p_2k^2 + \cdots)/(q_0 + q_1k + q_2k^2 + \cdots).$$

Show how the complete value of u might be found and why the special solution alone is needed in many practical problems.

Compare the results of applying the operation Z and the operation $(A + B\phi)/(C + D\phi)$, where

$$A \equiv (p_0 - p_2k^2 + p_4k^4 - \cdots),\ B \equiv (p_1 - p_3k^2 + p_5k^4 - \cdots),$$
$$C \equiv (q_0 - q_2k^2 + q_4k^4 - \cdots),\ D \equiv (q_1 - q_3k^2 + q_5k^4 - \cdots),$$

to $M \cdot \sin(kt + e)$.

Show that $(a + b\phi) \cdot [M \cdot \sin(kt + \alpha)]$

$$= M\sqrt{a^2 + b^2k^2} \cdot \sin[kt + \alpha + \tan^{-1}(bk/a)]$$

or
$$M\sqrt{a^2 + b^2k^2} \cdot \sin(kt + \delta),$$

where $\tan\delta = (bk\cos\alpha + a\sin\alpha)/(a\cos\alpha - bk\sin\alpha)$,

and that a special value of

$$\left(\frac{a + b\phi}{c + d\phi}\right) \cdot [M \cdot \sin(kt + \alpha)]$$

is $M\dfrac{\sqrt{a^2 + b^2k^2}}{\sqrt{c^2 + d^2k^2}} \cdot \sin[kt + \alpha + \tan^{-1}(bk/a) - \tan^{-1}(dk/c)]$.

This symbolic notation is treated at length in Forsyth's *Treatise on Differential Equations* and in Perry's *Calculus for Engineers.*

368. Prove that if ϕ stands for the operation D_t,

$$(a + b\phi + c\phi^2)[M \cdot \sin(kt + \alpha)] = M\sqrt{(a - ck^2)^2 + b^2k^2} \cdot \sin(kt + \alpha + \lambda),$$

where
$$\tan\lambda = bk/(a - ck^2),$$

and that a special value of

$$(l + m\phi + n\phi^2)^{-1}[M \cdot \sin(kt + \alpha)]$$

is
$$M \cdot \sin(kt + \alpha - \mu)/\sqrt{(l + nk^2)^2 + m^2k^2},$$

where $\tan\mu = mk/(l - nk^2)$. Hence show that a special value of

$$\left(\frac{a + b\phi + c\phi^2}{l + m\phi + n\phi^2}\right) \cdot [M \cdot \sin(kt + \alpha)]$$

is
$$M\frac{\sqrt{(a - ck^2)^2 + b^2k^2}}{\sqrt{(l - nk^2)^2 + m^2k^2}} \cdot \sin(kt + \alpha + \lambda - \mu).$$

369. If ϕ were an algebraic quantity, the expression

$$\frac{1}{\phi^2+(a+b)\phi+ab}$$

would be equivalent to

$$\frac{1}{b-a}\left(\frac{1}{\phi+a}-\frac{1}{\phi+b}\right).$$

Are these two operations equivalent when $\phi \equiv D_t$ and when a special value suffices?

370. If an electromotive force $E \equiv E_m \sin pt$ be applied to a circuit consisting of a coil of resistance r and inductance L, in series with a condenser of capacity K, we have the equation $E - L \cdot D_t C - Q/K = rC$, where $C = D_t Q$. Show that this equation can be written in the form

$$C = (r + L\phi + 1/K\phi)^{-1} E = K\phi E/(1 + rk\phi + Lk\phi^2)$$

and write down its solution in the form needed for practical use. Treat in the same manner several of the equations of Section 86.

371. Two circuits, s_1 and s_2, have resistances r_1, r_2, inductances L_1, L_2, capacities K_1, K_2, and a mutual inductance M. They contain variable electromotive forces E_1, E_2 and carry currents C_1, C_2. Show that if

$$R_1 \equiv r_1 + L_1\phi + 1/K_1\phi,$$
$$R_2 \equiv r_2 + L_2\phi + 1/K_2\phi,$$

where ϕ represents the operation D_t,

$$C_1 = (R_2E_1 - M\phi E_2)/(R_1R_2 - M^2\phi^2),$$
$$C_2 = (R_1E_2 - M\phi E_1)/(R_1R_2 - M^2\phi^2).$$

If the capacities of the circuits are negligible, we are to put $R_1 \equiv r_1 + L_1\phi$, $R_2 \equiv r_2 + L_2\phi$ in these results.

If $E_2 = 0$, $K_1 = \infty$, $K_2 = \infty$;

$$C_1 = (r_2 + L_2\phi)E_1/[r_1r_2 + (r_2L_1 + r_1L_2)\phi + (L_1L_2 - M^2)\phi^2],$$

and $C_2 = -M\phi E_1/[r_1r_2 + (r_2L_1 + r_1L_2)\phi + (L_1L_2 - M^2)\phi^2]$.

Write down the solutions of these equations and compare them with the results given in Section 87.

372. Show that if C_1, C_2 are the currents in the primary and secondary of a transformer, and if the secondary circuit has no capacity and contains no internal applied electromotive force, $M\phi C_1 + (r_2 + L_2\phi) C_2 = 0$ or $C_2 = -M\phi C_1/(r_2 + L_2\phi)$, and if $C_1 = C_m \sin(pt - a)$,

$$C_2 = \frac{-MpC_m}{\sqrt{L_2^2p^2 + r_2^2}} \cdot \sin\left(pt - a + \frac{\pi}{2} - \tan^{-1}L_2p/r_2\right).$$

If, as is often the case in practice, r_2 is small compared with L_2p, we have, approximately,

$$\begin{aligned} C_2 &= -MC_m \cdot \sin(pt - a) \cdot /L_2 \\ &= MC_m \cdot \sin(pt - a - \pi) \cdot /L_2. \end{aligned}$$

In the general case C_2 and C_1 are not zero at the same instant.

373. An electrodynamometer consists essentially of two coils, one fixed and the other movable. The movable coil is furnished with an index which moves on a fixed scale, and the readings are to be considered equal to the product of the strengths of the steady currents C_1, C_2 in the two coils. If, however, these currents alternate rapidly, the readings are proportional to the average value of C_1C_2. Assuming that

$$\int \sin pt \cdot \sin qt \cdot dt$$
$$= \sin(p - q)t \cdot /2(p - q) - \sin(p + q)t \cdot /2(p + q),$$
$$\int \sin^2 pt\, dt = (pt - \sin pt \cdot \cos pt)/2p,$$

and that $C_1 = m \cdot \sin pt$, $C_2 = n \cdot \sin qt$, show that the reading is zero when p and q are not equal. What is it when $p = q$? Plot a curve which shall show the readings for different values of a, when $C_1 = m \cdot \sin pt$, $C_2 = n \cdot \sin(pt - a)$, assuming that

$$\int \sin pt \cdot \cos pt\, dt = -\cos 2pt \cdot /4p.$$

If $a = \frac{1}{2}\pi$, the reading is zero.

374. A function $z \equiv f(t)$ may be represented in polar coördinates at any instant by a point P, the distance of which from the origin O is equal to the numerical value of z, while the vectorial angle XOP is equal to pt, where p is any convenient constant. The plane path traced out by P during any time interval shows the march of z during the interval. If z is either $a \cdot \cos(pt - \delta)$ or $a \cdot \sin(pt - \delta)$, the path of P is a circumference of diameter a passing through the origin: the vectorial angle of the centre of the circumference is δ in the first case and $\delta + \frac{1}{2}\pi$ in the second.

If z is known to be a simple sine or a simple cosine function of frequency $p/2\pi$, it is completely determined when the vector OP_0, which represents the diameter of the circumference, is given. If the plane of the diagram were the ordinary complex plane, P_0 would represent the complex quantity $z_0 \equiv x_0 + jy_0$, where x_0 and y_0 are the horizontal and vertical projections of the diameter of the circumference, and j the imaginary unit; and it is often convenient, as Steinmetz has shown in a series of remarkable papers, to represent the harmonic function z by the quantity z_0. With this understanding of the meaning of the sign of equality, we may write in general $z = x_0 + y_0 j$: the modulus of z is given by the equation $|z| = \sqrt{x_0^2 + y_0^2}$.

Show that if $C' + j \cdot C''$ represents the current

$$C \equiv C_m \cdot \sin(pt - \delta),$$

where $\tan\delta = Lp/r$, in a simple circuit of resistance r, inductance L, reactance $x \equiv Lp \equiv 2\pi nL$, and impedance Z; the "electromotive force consumed by the resistance" is

$$rC = r(C' + j \cdot C''),$$

the "electromotive force produced by the reactance" is

$$jxC = jxC' - xC'',$$

and the electromotive force required to overcome the reactance is $xC'' - jxC'$, so that the applied electromotive force E is

$$(rC' + xC'') + j(rC'' - xC') \equiv e_r' + e_x \cdot j \equiv (r - jx)\, C,$$

and the impedance is $(r - jx)$.

Write down an expression in complex form for the impedance of a simple circuit made up of a number of coils of resistances r_1, r_2, r_3, $\cdots$, and inductances L_1, L_2, L_3 in series.

Show that if an electromotive force, $E_m \cdot \cos pt$, be applied to a simple circuit of resistance r and inductance L which contains a condenser of capacity k, the impedance Z has the form $r - j(x - x')$, where $x \equiv Lp$ and $x' \equiv 1/kp$.

375. A long straight wire parallel to the x axis, of resistance r and self-inductance L per unit length, is covered by a thin layer of insulation the outside of which is kept at potential zero. The capacity of the cable per unit length is k, and the rate of leakage through the insulation of a point where the wire is at potential V is λV per unit of length. Show that if C is the current in the wire,

$$D_x C + \lambda V + k \cdot D_t V = 0, \quad D_x V + L \cdot D_t C + rC = 0;$$

or,

$$D_x^2 V - Lk \cdot D_t^2 V - (L\lambda + rk)\, D_t V - r\lambda V = 0,$$

$$D_x^2 C - Lk \cdot D_t^2 C - (L\lambda + rk)\, D_t C - r\lambda C = 0.$$

[Heaviside, *Electrical Papers,* Vol. I, XX; Poincaré, *Comptes Rendus,* 1893; Picard, *Comptes Rendus,* 1894; Boussinesq, *Comptes Rendus,* 1894; Bedell and Crehore, *Alternating Currents;* Webster, *Electricity and Magnetism;* Pupin, *Transactions of the American Mathematical Society,* 1900; *The Electrical World and Engineer,* October, 1901, and February, 1902.]

376. Two circuits s_1, s_2, which have self-inductances L_1, L_2, and a mutual inductance M, carry currents C_1, C_2 in a magnetic field due to these currents only. The first circuit, which is rigid, contains a generator of constant electromotive force E_1; the second, which is deformable, contains no generator, so that

$E_1 - d(L_1C_1 + MC_2)/dt = r_1C_1, \quad -d(MC_1 + L_2C_2)/dt = r_2C_2.$ Show that if one (x) of the generalized coördinates which define the conformation of the second circuit receives the increment dx during the time dt, so that L_2, M, C_1, C_2 are changed while L_1 remains constant, the work dW done by the electromagnetic forces is $\frac{1}{2}C_2^2 \cdot dL_2 + C_1C_2 \cdot dM$, and the change dT in the electrokinetic energy is

$$\tfrac{1}{2}C_2^2 \cdot dL_2 + L_1C_1 \cdot dC_1 + MC_1 \cdot dC_2 \\ + MC_2 \cdot dC_1 + C_1C_2 \cdot dM + L_2C_2 \cdot dC_2.$$

Show that the equation $(-dp_2 = r_2C_2 \cdot dt)$ yields

$$r_2C_2^2 \cdot dt + C_2L_2 \cdot dC_2 + C_2^2 \cdot dL_2 + MC_2 \cdot dC_1 + C_1C_2 \cdot dM = 0,$$

and that the energy $(C_1^2rdt + C_1dp_1)$ furnished during the interval dt by the generator in the first circuit is equal to $dW + dT$ plus the energy dissipated in heat in the two circuits. If C_2 is originally zero, the expressions for dW and dT are much simplified. In any given case dx/dt is virtually determined by the mechanical equation of motion of the moving parts of s_2; its value will evidently be greater or smaller, other things being equal, according as the electromagnetic forces are assisted or opposed by external forces. L_2, M are to be regarded as given functions of x and other variables which do not here enter, and dL_2/dt, dM/dt can be written $D_xL_2(dx/dt)$ and $D_xM(dx/dt)$. The mechanical equation and the first two equations of this problem form a set which completely determine x, C_1, C_2 as functions of the time.

If a circuit s is threaded by $M\Phi$ lines from a magnetic shell of strength Φ, $E \cdot dt - d(M\Phi + LC) = rC \cdot dt$ and, in the general case, M, Φ, L, and C are all functions of the time.

377. If μ is the mass of the slider AB in Fig. 69, and if a constant force X be applied to AB towards the right, if $DG \equiv l$, $GB \equiv x$, $v \equiv D_tx$, we have

$$\mu \cdot D_tv = \tfrac{1}{2}C^2 \cdot D_xL + CHl + X,$$
$$E \cdot dt - d(LC + Hlx) = rC \cdot dt.$$

Show that if we can neglect the effect of the field due to the current in $DABG$, and if the change in r during the motion is inappreciable,

$$C = [E/r + X/Hl]e^{-\lambda t} - X/Hl,$$

$$v = [rX/H^2l^2 + E/Hl]\,(1 - e^{-\lambda t}),$$

where $\lambda \equiv H^2l^2/\mu r.$

If X is positive, the current changes sign when

$$H^2l^2t = \mu r \cdot \log[(EHl + rX)/rX].$$

378. A condenser made of two circular pieces of tin foil, each 28.58 centimetres in diameter, separated by a plate of plane glass of inductivity 6, $\frac{1}{4}$ of a centimetre thick, is discharged by means of a piece of non-magnetic wire 1 metre long, bent into the form of a nearly complete circle. The resistance of the circuit is 0.001 ohm, and its self-induction 1000 electromagnetic absolute units. Assuming that the farad is equivalent to 9×10^{11} electrostatic absolute units of capacity, show that the discharge will be oscillatory with a period of about 2.3×10^{-7} seconds. The time constant, $2L/R$, is 0.002 second. The amplitude would be reduced to $\frac{1}{1000}$ of its initial value in about 0.014 second, and to $\frac{1}{1000000}$ of this value in about 0.028 second.

379. A spherical shell of copper of small uniform thickness and of radius a is in a magnetic field of uniform intensity H. Show that the work required to withdraw it instantaneously from the field is $\frac{1}{4}H^2a^3$. [M. T.]

380. An alternating electric current $C \cdot \cos pt$ is made to flow along a straight wire of uniform circular section. Prove that the current strength at a distance r from the axis of the wire is given by the real part of $Cka \cdot J_0(kr) \cdot e^{pti}/[2\pi a^2 \cdot J_1(ka)]$, where a is the radius of the wire, ρ its specific resistance, and $k \equiv (1 - i)\sqrt{2\pi p}/\sqrt{\rho}$. [M. T.]

381. If x is a function continuous within the extremely short time interval T, and if τ represents any instant during the interval, $\int_0^\tau x \cdot dt$ is small and $\int_0^T dt \int_0^\tau x \cdot dt$ much smaller.

A wire circuit of conductivity C and self-inductance L is situated in the field of a magnetic system which undergoes a disturbance of impulsive character such that at the end it has returned to its initial state. Prove that if M denote the change in the induction through the circuit due to the field at any instant during the disturbance, and if N is the time integral of M taken throughout the whole time of disturbance, then the induced current at any subsequent time t is $Ne^{-t/CL} \cdot / CL^2$. [M. T.]

382. Show that if the branches p, q, r, and s of the Wheatstone net have self-inductances L_p, L_q, L_r, L_s, and contain condensers which have capacities k_p, k_q, k_r, k_s, and if the current C in the main circuit is a given function of the time, the currents in the other branches are to be found from the equations

$$C_r + C_s = C,\ C_p = C_r + C_g,\ C_q = C_s - C_g,$$

$$L_p \cdot D_t^2 C_p + p \cdot D_t C_p$$
$$+ C_p / k_p - L_q \cdot D_t^2 C_q - q \cdot D_t C_q - C_q / k_q + g \cdot D_t C_g = 0,$$

$$L_r \cdot D_t^2 C_r + r \cdot D_t C_r$$
$$+ C_r / k_r - L_s \cdot D_t^2 C_s - s \cdot D_t C_s - C_s / k_s - g \cdot D_t C_g = 0.$$

(1) Prove that if C_x' is the current in any branch, x, when $C = F(t)$, and C_x'' the current in the same branch when $C = f(t)$, $C_x' + C_x''$ will be the current when $C = F(t) + f(t)$.

(2) Show that if $C_x' + i \cdot C_x''$ is the value of C_x obtained from these equations when $C = F(t) + i \cdot f(t)$, C_x' would correspond to $C = F(t)$ and C_x'' to $C = f(t)$.

(3) Show that if $C \equiv A \cdot e^{\lambda t}$, the equations are satisfied, when the coefficients are properly determined, by the values

$$C_p = A_p \cdot e^{\lambda t},\ C_q = A_q \cdot e^{\lambda t},\ C_r = A_r \cdot e^{\lambda t},\ C_s = A_s \cdot e^{\lambda t},\ C_g = A_g \cdot e^{\lambda t},$$

and that if $b_x \equiv 1 + \lambda k_x(x + \lambda \cdot L_x)$,

$$(b_p k_q + b_q k_p) A_p + g\lambda k_p k_q A_g = b_q k_p A,$$
$$(b_r k_s + b_s k_r) A_p - (b_r k_s + b_s k_r + g\lambda k_r k_s) A_g = b_s k_r A.$$

Find A_g.

(4) Show that if the condensers are all removed, if

$$C \equiv A \cdot e^{\lambda t},$$

and if

$$a_x \equiv x + \lambda \cdot L_x;\ A_g = A(a_r \cdot a_q - a_p \cdot a_s) / [(a_p + a_q)(a_r + a_s) + g(a_p + a_q + a_r + a_s)].$$

(5) If $\lambda \equiv mi$, $C = A(\cos mt + i \sin mt)$, A_g has the form $M + Ni$, and C_g the form

$$(M + Ni)(\cos mt + i \sin mt) = (M \cdot \cos mt - N \cdot \sin mt) + i(M \cdot \sin mt + N \cdot \cos mt).$$

If the condensers are all removed, and if $C = A \cdot \cos mt$, what is the condition that no current shall pass through g?

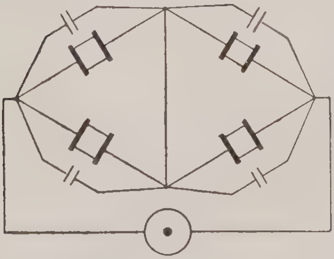

FIG. 134.

383. Show that if (1) the branches p, q, r, and s of the Wheatstone net have self-inductances L_p, L_q, L_r, L_s, and are in parallel with branches of negligible resistance having capacities k_p, k_q, k_r, k_s, if (2) the current C in the main circuit (Fig. 134) is a given function of the time, and if (3) the currents in the branch x of the net and in the condenser circuit parallel to it are denoted by C_x and C_x' respectively, the currents are to be determined with the help of the equations

$$x \cdot C_x = -L_x \cdot D_t C_x + Q_x / k_x \text{ or } C_x' = k_x(x \cdot D_t C_x + L_x \cdot D_t^2 C_x),$$
$$C_r + C_r' + C_s + C_s' = C, \quad C_p + C_p' = C_g + C_r + C_r',$$
$$C_q + C_q' = C_s + C_s' - C_g,$$
$$p \cdot C_p + g \cdot C_g - q \cdot C_q = -L_p \cdot D_t C_p + L_q \cdot D_t C_q,$$

and $$r \cdot C_r - s \cdot C_s - g \cdot C_g = -L_r \cdot D_t C_r + L_s \cdot D_t C_s.$$

If the results of applying the operations

$$[1 + k_x(x \cdot D_t + L_x \cdot D_t^2)],\ [x + L_x \cdot D_t] \text{ to } C_x$$

be denoted by $\phi(C_x)$, $\psi(C_x)$ respectively, these equations yield the five equations which follow:

$$\begin{aligned}
\phi(C_r) + \phi(C_s) &= C,\\
\phi(C_p) - \phi(C_r) - C_g &= 0,\\
\phi(C_q) - \phi(C_s) + C_g &= 0,\\
\psi(C_r) - \psi(C_s) - g \cdot C_g &= 0,\\
\psi(C_p) - \psi(C_q) + g \cdot C_g &= 0.
\end{aligned}$$

Show that if $C = A \cdot e^{\lambda t}$, these equations are satisfied, when the coefficients are properly determined, by the values $C_p = A_p \cdot e^{\lambda t}$, $C_q = A_q \cdot e^{\lambda t}$, $C_r = A_r \cdot e^{\lambda t}$, $C_s = A_s \cdot e^{\lambda t}$, $C_g = A_g \cdot e^{\lambda t}$, and that if $a_x \equiv x + L_x \cdot \lambda$, $b_x = 1 + k_x \lambda (x + L_x \cdot \lambda)$,

$$A_g = \frac{A(a_q a_r b_p b_s - a_p a_s b_q b_r)}{(a_p b_q + a_q b_p)(a_r b_s + a_s b_r) + g[b_p b_q (a_r b_s + a_s b_r) + b_r b_s (a_p b_q + a_q b_p)]}.$$

If A is real, and if $\lambda = mi$, where m is real,

$$C = A(\cos mt + i \cdot \sin mt),$$

A_g is of the form $M + Ni$, and C_g of the form

$$(M + Ni)(\cos mt + i \cdot \sin mt)$$

or $$(M \cdot \cos mt - N \cdot \sin mt) + i(M \cdot \sin mt + N \cos mt).$$

The real part of C_g is of the form $\sqrt{M^2 + N^2} \cdot \cos(mt - \delta)$, where $\tan \delta = -N/M$. Prove that this would be the value of C_g if the value of C were $A \cdot \cos mt$.

Show that if $C = A \cdot \cos mt$, if the condensers are all removed, and if $L_q = L_s = 0$, C_g will be zero for all values of m if $L_p / L_r = q/s = p/r$.

Prove that if $C = A \cdot \cos mt$, if the inductances are negligible, and if $k_q = k_s = 0$, C_g will be zero for all values of m if $k_p / k_r = s/q = r/p$.

Prove that if all the L's and k's except k_p and L_s are zero, C_g will be zero if, and only if, $L_s = qrk_p = psk_p$.

Show how to find the value of C_g if $C = A \cdot \sin mt$.

384. An infinite mass of metal has one plane face, which is the yz plane. At the time zero, uniform currents parallel to the z axis are induced in the plane by a sudden change in the magnetic field, which after the change remains constant. It is evident that u and v will remain zero and that w is a function of x only, so that (Section 88) $4\pi\mu\lambda \cdot D_t w = D_x{}^2 w$ and $w = A \cdot e^u / t^{\frac{1}{2}}$, where $u = -\pi\mu\lambda x^2 / t$. Show that w will have its maximum value at a distance x_0 from the plane face at a time $t = 2\pi\mu\lambda x_0{}^2$ and that this value is $A / (x\sqrt{2\pi\mu\lambda e})$. When t is large, w has nearly the same value for all moderate values of x. Assuming that for copper $\mu\lambda = 1/1600$ and for iron $1/10$, show that the maximum current will be attained at a depth of 16 centimetres in copper and 1.26 centimetres in iron after 1 second.

385. An infinite conductor has one plane face (the yz plane), but is otherwise unlimited. Periodic currents parallel to the z axis and independent in intensity of y and z are induced in this conductor by some cause on the negative side of the yz plane. Since u and v are zero, w must (Section 88) satisfy the equation $4\pi\lambda\mu \cdot D_t w = D_x{}^2 w$, and of this equation $Ce^{lt} \cdot e^{-nx}$, where $l = k^2 i / 4\pi\lambda\mu$, $n^2 = k^2 i$, $n = k(1+i)/\sqrt{2}$, is a special solution. The real part of a complex solution of this equation is itself a special solution; and if we assume that $w = C_m \cos pt$, when $x = 0$, we learn that

$$w = C_m e^{-x\sqrt{2\pi\lambda\mu p}} \cdot \cos(pt - x\sqrt{2\pi\lambda\mu p}),$$

a simple harmonic expression the amplitude of which is $C_m e^{-2\pi x\sqrt{\lambda\mu f}}$, where f is the number of alternations per second. Show that this amplitude has $1/m$th of its surface value at the depth $\log m / (2\pi\sqrt{\lambda\mu f})$. Show that if the frequency is 100, the amplitudes 1 centimetre from the surface will be 0.208 C_m in the case of copper, and 0.0000000024 C_m if the conductor is made of iron. Show also that if the frequency is 10,000 the amplitude in the case of copper at a point 1 centimetre from the surface would be $0.00000015\, C_m$.

386. Show that if U, V, W are the components of a vector taken at every point in the direction in which the orthogonal curvilinear coördinates u, v, w increase most rapidly, the components of the curl of the vector are

$$\begin{aligned} K_u &= h_v h_w [D_v(W/h_w) - D_w(V/h_v)], \\ K_v &= h_w h_u [D_w(U/h_u) - D_u(W/h_w)], \\ K_w &= h_u h_v [D_u(V/h_v) - D_v(U/h_u)]. \end{aligned}$$

A very interesting proof of Stokes's Theorem in curvilinear coördinates has been given by Prof. A. G. Webster in the *Bulletin of the American Mathematical Society* for 1898.

387. The whole induction flux through a linear circuit, s_k, is equal to the line integral, I_k, of the tangential component of a vector potential of the induction taken around the circuit, so that the electrokinetic energy, T, of a set of circuits $s_1, s_2, s_3, \cdots$, which carry currents $C_1, C_2, C_3, \cdots$, is

$$\tfrac{1}{2}\sum C_k I_k = \tfrac{1}{2}\sum C_k \int [F_x \cdot \cos(x, s_k) + F_y \cdot \cos(y, s_k) + F_z \cdot \cos(z, s_k)]\, ds_k.$$

If the linear circuits are the filaments of a massive conductor in which the components of the current are u, v, w, and if S_k is the area, at any place, of the cross-section of the filament s_k; $C_k \cdot \cos(x, s_k) = u \cdot S_k$, $C_k \cdot \cos(y, s_k) = v \cdot S_k$, $C_k \cdot \cos(z, s_k) = w \cdot S_k$. Show that we may write

$$T = \tfrac{1}{2}\iiint (F_x \cdot u + F_y \cdot v + F_z \cdot w)\, d\tau,$$

where the integration is to be extended over all space where currents exist.

ADDITIONAL ANSWERS TO PROBLEMS.

CHAPTER I.

(1) $X=\frac{113032}{274625}$, $Y=\frac{96501}{274625}$, whence $R=0.541+$; $\theta=40^\circ\,29'+$.

(2) $R=m\lambda$, $\cos\alpha=\frac{1}{a^2\lambda}$, $\cos\beta=\frac{1}{b^2\lambda}$, $\cos\gamma=\frac{1}{c^2\lambda}$, where $\lambda=\sqrt{\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}}$.

(5) If $2l$ be the length of the wire, and if the axis of the wire be taken for the axis of abscissas with the origin at the middle point, the required equation is

$$2c^2y^2-1=\frac{l^2-x^2-y^2}{[(l-x)^2+y^2]^{\frac{1}{2}}\cdot[(l+x)^2+y^2]^{\frac{1}{2}}}.$$

(6) If the radius of the earth be taken as 3960 miles, and the mass of a cubic foot of water as 62.5, one poundal is equivalent to 952 million attraction units approximately.

(8) If c be the distance of the point P from the centre of the sphere, the required attraction is $m\left[\frac{1}{c^2}-\frac{1}{8(c\pm r)^2}\right]$, if P is without the solid; $\frac{m}{8r^2}$, if P is within the cavity, and

$$\frac{m}{8r^3}\left[c-\frac{r^3}{(c+r)^2}\right],$$

if P is a point in the mass of the solid.

(9) The attraction of a hemisphere of radius R at a point P facing the flat side of the hemisphere and lying on the perpendicular to this side erected at the centre O is

$$\frac{M}{2a^2R^3}[R^3 - 2a^3 + (2a^2 - R^2)\sqrt{R^2 + a^2}], \text{ where } a = OP.$$

The attraction of the other hemisphere may be found by subtracting this quantity from the attraction due to the whole sphere of which this hemisphere is a part. See § 9.

(10) (*a*) That the density varies inversely as the distance from the centre.

(*b*) $$\frac{M}{4\pi R^2}\left[\frac{2}{r} + \frac{2R}{981nr} - \frac{1}{327n}\right].$$

(12) Here d is supposed to be greater than a.

(15) The attraction is 34.9, and its line of action makes an angle of 1° 49′ with the line joining the centre of the sphere with the point in question.

CHAPTER II.

(7) That the force is constant.

(9) $V_{0-1} = \frac{2}{3}\pi\rho(18 - c^2)$; $V_{1-2} = \frac{2}{3}\pi\rho\left(15 + \frac{2}{c}\right)$;

$$V_{2-3} = \frac{2}{3}\pi\rho\left(27 - \frac{14}{c} - c^2\right); \; V_{1-\infty}\frac{2}{3}\pi\rho\frac{40}{c}.$$

(11) Yes; 1.46 –.

(12) If $3 > r$, $V = \pi\mu\left(\frac{298}{3} - \frac{2r^2}{3}\right)$;

if $4 > r > 3$, $V = \pi\mu\left(\frac{244}{3} + \frac{36}{r}\right)$;

if $5 > r > 4$, $V = \pi\mu\left(\frac{500}{3} - \frac{r^3}{3} - \frac{220}{r}\right)$;

if $r > 5$, $V = \pi\mu\frac{405}{r}$.

(15) No.

(20) (1) About 1,830,000 tons of 2000 pounds each.

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